Mathematical Analysis supported by wxMaxima

Rudolf Blaško
University of Žilina

Project: Innovative Open Source Courses for Computer Science

31. 5. 2021

Co-funded by the
Erasmus+ Programme
of the European Union

Contents

1. Introduction to wxMaxima
2. Real functions
3. Differential calculus
4. Indefinite integral
5. Definite integral

Innovative Open Source Courses for Computer Science

This teaching material was written as one of the outputs of the project
“Innovative Open Source Courses for Computer Science”,
funded by the Erasmus+ grant no. 2019-1-PL01-KA203-065564.
The project is coordinated by West Pomeranian University of Technology in Szczecin (Poland)
and is implemented in partnership with Mendel University in Brno (Czech Republic)
and University of Žilina (Slovak Republic).
The project implementation timeline is September 2019 to December 2022.

Innovative Open Source Courses for Computer Science

Project was implemented under the Erasmus+.
Project name: "Innovative Open Source courses for Computer Science curriculum"
Project no.: 2019-1-PL01-KA203-065564
Key Action: KA2 – Cooperation for innovation and the exchange of good practices
Action Type: KA203 – Strategic Partnerships for higher education

Consortium: Zachodniopomorski uniwersytet technologiczny w Szczecinie
Mendelova univerzita v Brně
Žilinská univerzita v Žiline

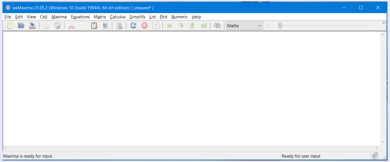
Co-funded by the
Erasmus+ Programme
of the European Union

Introduction to wxMaxima

Mathematical Analysis supported by wxMaxima

Basic terms

- wxMaxima is a document based interface for the computer algebra system Maxima.
- wxMaxima is distributed under the GPL license.
- The program can be compiled in various OS (Windows, GNU/Linux, MacOS X, . . .).
- xMaxima is a graphical interface for Maxima, written in Tcl/Tk.
- Maxima is one of the Open Source programs with open source code.
- A precompiled program for GNU/Linux and Windows is available free of charge on the SourceForge website `https://sourceforge.net/projects/maxima/files/`.
- After starting the wxMaxima environment, a menu window will appear on the screen.
- Below the menu is a space where we can enter commands and where outputs appear.



wxMaxima 21.05.2 (Windows 10 (build 19044), 64-bit edition) [unsaved*]

Maxima is ready for input.

Ready for user input

Basic terms

- We enter commands on separate lines (input lines).

  Their execution is ensured by simultaneously pressing the `Shift` keys and `Enter` or by clicking on in the menu icon ➡ (Send the current cell to maxima).

- Input lines are listed with (%i1).

- Output lines are listed with (%o1).

- The numbers for the input line and the corresponding output line are identical and based on this number, we can refer to the content of these lines.

```
(%i1) First input line.
(%o1) First output line.
(%i2) Second input line.
(%o2) Second output line.
```

Basic terms

- The commands are executed on new separate lines (output lines).
- Commands on input lines can be terminated with the symbol `;` or the `$` symbol, which suppresses the display of the corresponding output.

```
(%i1) solve(0=x+2,x);
(%o1) [x = −2]
(%i2) %i1;
(%o2) solve(0 = x + 2, x)
(%i3) %o1;
(%o3) [x = −2]
```

Basic terms

We can save the output in various shapes and then use it in other programs.

(%o3) $[x=-\frac{2}{3},x=0]$

Output (%o3) from the previous window we can:

- Copy `Ctrl C` and `Ctrl V`, respectively copy as text (can be used eg. for MSWord equation editor): `x=-2/3,x=0`,
- Copy as LaTeX `\[x=-\frac{2}{3}\operatorname{,}x=0\]`,
- Copy as MathML, image, RTF, SVG…

The wxMaxima environment has a well-designed user help, which can be found in the `Help` menu. You can also open Help by pressing the `F1` key.

You can also find the manual on the website `https://maxima.sourceforge.io/docs/manual/maxima_369.html`.

Basic terms

We can save the output in various shapes and then use it in other programs.

(%o3) $[x=-\frac{2}{3},x=0]$

Output (%o3) from the previous window we can:

- Copy `Ctrl C` and `Ctrl V`, respectively copy as text (can be used eg. for MSWord equation editor): `x=-2/3,x=0`,
- Copy as LaTeX `\[x=-\frac{2}{3}\operatorname{,}x=0\]`,
- Copy as MathML, image, RTF, SVG…

The wxMaxima environment has a well-designed user help, which can be found in the `Help` menu. You can also open Help by pressing the `F1` key.

You can also find the manual on the website `https://maxima.sourceforge.io/docs/manual/maxima_369.html`.

Basic terms

We can save the output in various shapes and then use it in other programs.

(%o3) $[x = -\frac{2}{3}, x = 0]$

Output (%o3) from the previous window we can:

- Copy `Ctrl C` and `Ctrl V`, respectively copy as text (can be used eg. for MSWord equation editor): `x=-2/3,x=0`,
- Copy as LaTeX `\[x=-\frac{2}{3}\operatorname{,}x=0\]`,
- Copy as MathML, image, RTF, SVG…

The wxMaxima environment has a well-designed user help, which can be found in the `Help` menu. You can also open Help by pressing the `F1` key.

You can also find the manual on the website `https://maxima.sourceforge.io/docs/manual/maxima_369.html`.

Basic Commands

- Using `apropos` we find out the exact name of the command using part of its name.

```
(%i1)  apropos("plot")
(%o1)  [barsplot,boxplot,contour_plot,get_plot_option,gnuplot,...
```

- Command `describe` prints a description of the entered command.

```
(%i1)  describe(plot2d)$
       -- Function:  plot2d
       plot2d (<expr><,<range_x><,<options><)
       plot2d (<expr_<>=<expr_<>,<range_x><,<range_y><,<options><)
       plot2d ([parametric,<expr_x><,<expr><_y,<range><],<options><)
       plot2d ([discrete,<points><],<options><)
       plot2d ([contour,<expr><],<range_x><,<range_y><,<options><)
       plot2d ([<type_<>,...,<type_n><],<options><)
       There are 5 types of plots that can be plotted by 'plot2d':
             1.  Explicit functions.  'plot2d' ...
       ...
```

Basic Commands

- Expressions are entered using the usual characters of operations, sessions and functions.
- Arguments of functions and commands are in parentheses.
- Multiplication symbol `*` must be entered!
- The exponentiation is specified by the character `^` or the pair `**`.
- Symbol `:` is used to assign a value to the right of the expression to the left.
- The following commands solve the equation $2x + 3x^2 = 0$ with unknown variable x.

```
(%i1) a:2$ b:3$ solve(a*x+b*x^2=0,x);
```

(%o1) $[x = -\frac{2}{3}, x = 0]$

- With the `kill` command we can remove variables with all their assignments and properties from memory.

```
(%i1) kill(a,b)
        /* removes all bindings from the arguments a,b */
(%i2) kill(all) /* removes all items on all infolists */
```

beerb@frcatel.fri.uniza.sk http://frcatel.fri.uniza.sk/users/beerb

Basic Commands

- Expressions are entered using the usual characters of operations, sessions and functions.
- Arguments of functions and commands are in parentheses.
- Multiplication symbol `*` must be entered!
- The exponentiation is specified by the character `^` or the pair `**`.
- Symbol `:` is used to assign a value to the right of the expression to the left.
- The following commands solve the equation $2x+3x^2=0$ with unknown variable x.

```
(%i1) a:2$ b:3$ solve(a*x+b*x^2=0,x);
```

(%o1) $[x=-\frac{2}{3}, x=0]$

- With the `kill` command we can remove variables with all their assignments and properties from memory.

```
(%i1) kill(a,b)
       /* removes all bindings from the arguments a,b */
(%i2) kill(all) /* removes all items on all infolists */
```

Basic Commands

- Expressions are entered using the usual characters of operations, sessions and functions.
- Arguments of functions and commands are in parentheses.
- Multiplication symbol `*` must be entered!
- The exponentiation is specified by the character `^` or the pair `**`.
- Symbol `:` is used to assign a value to the right of the expression to the left.
- The following commands solve the equation $2x + 3x^2 = 0$ with unknown variable x.

```
(%i1)  a:2$ b:3$ solve(a*x+b*x^2=0,x);
```

(%o1) $[x = -\frac{2}{3}, x = 0]$

- With the `kill` command we can remove variables with all their assignments and properties from memory.

```
(%i1)  kill(a,b)
        /* removes all bindings from the arguments a,b */
(%i2)  kill(all) /* removes all items on all infolists */
```

Basic Commands

- In the menu `View` and submenu `Display equations` we can change display output lines for shapes `in 2D` (default form), `as 1D ASCII` or `as ASCII Art`.
- You can also change the output settings with the command `set_display`.

```
(%i1) x/sqrt(x^2+1);set_display('none)$
```

(%o1) $\frac{x}{\sqrt{x^2+1}}$ `/* in 2D */`

```
(%i1) x/sqrt(x^2+1);set_display('ascii)$
```

(%o1) $x/\mathrm{sqrt}(x^2+1)$ `/* as 1D ASCII */`

```
(%i2) x/sqrt(x^2+1);set_display('xml)$
                x
(%o2) ----------          /* as ASCII Art */
                  2
      sqrt(x + 1)
```

Working with Numbers and Basic Constants

- Maxima can work with real numbers written in numerical or symbolic form.
- The way of writing real numbers can be set in the menu `Numeric` using the switch `Numeric Output` between numeric and symbolic display.
- The setting of the variable `numer` determines the method of displaying.
- By default, 16 digits (including the decimal point) are displayed.
- The display accuracy is defined by the variable `fpproc` and affects the display using `bfloat`. Output `float` always displays the same.
- By default, complex numbers are entered in algebraic form (`rectform`). They can be converted to trigonometric (exponential) form using the command `polarform`.

(%i1) `z:1+%i;`

(z) $i+1$

(%i2) `polarform(z)+rectform(z);`

(%o2) $\sqrt{2}\,e^{\frac{i\pi}{4}}+i+1$

Working with Numbers and Basic Constants

- Maxima can work with real numbers written in numerical or symbolic form.
- The way of writing real numbers can be set in the menu `Numeric` using the switch `Numeric Output` between numeric and symbolic display.
- The setting of the variable `numer` determines the method of displaying.
- By default, 16 digits (including the decimal point) are displayed.
- The display accuracy is defined by the variable `fpproc` and affects the display using `bfloat`. Output `float` always displays the same.
- By default, complex numbers are entered in algebraic form (`rectform`). They can be converted to trigonometric (exponential) form using the command `polarform`.

(%i1) `z:1+%i;`

(z) $i+1$

(%i2) `polarform(z)+rectform(z);`

(%o2) $\sqrt{2}\,e^{\frac{i\pi}{4}}+i+1$

Working with Numbers and Basic Constants

- Numeric constants e, π, i (imaginary unit) have the prefix `%`, i.e. `%e`, `%pi`, `%i`. They have the `%` prefix even if they are part of or the result of a calculations.
- Maxima has predefined constants `inf`, `minf` for real infinite ∞, $-\infty$.
- Maxima has predefined constants `infinity` for complex infinity.
- Logical constants `true` and `false` they represent truth and untruth.

```
(%i1) %pi+%i+%e;
(%o1) π + %i+%e
(%i2) [minf,inf];
(%o2) [−∞, ∞]
(%i3) infinity;
(%o3) infinity
```

Assignments and Functions

- Maxima contains many more functions than standard programming languages.
  These are not only the functions themselves, but also various functions for their support.
- The `:` operator we use to assign values or expressions to variables.
- We define functions using the assignment `:=`.

(%i1) `f(x):=x^2+2*x+3;`

(%o1) $f(x) := x^2 + 2x + 3$

(%i6) `f(x);f(y);f(x+1);`
`f(-2);f(1);`

(%o2) $x^2 + 2x + 3$

(%o3) $y^2 + 2y + 3$

(%o4) $(x + 1)^2 + 2(x + 1) + 3$

(%o5) 3

(%o6) 6

Working with Expressions

Many times we need to change the conditions only locally for a particular calculation without to change global settings. For this purpose, Maxima has a very efficient `ev` command.

- The command `ev` allows defining a specific environment within a single command.
- After entering the command `ev(a,b1,b2,..., bn)` the expression `a` is evaluated if the conditions `b1`, `b2`, . . . , `bn` are met.
- These conditions can be equations, assignments, functions, switches (logical settings).

The example shows an example of solving a quadratic equation using the command `solve`.

- Variables `a`, `b`, `c` after executing the command `ev` they do not have values assigned.

(%i1) `ev(solve(a*x^2+b*x+c=0,x),a:2,b:-1,c=-3);`

(%o1) $[x=\frac{3}{2}, x=-1]$

(%i2) `solve(a*x^2+b*x+c=0,x);`

(%o2) $[x=-\frac{\sqrt{b^2-4ac}+b}{2a}, x=\frac{\sqrt{b^2-4ac}-b}{2a}]$

Working with Expressions

We can substitute expressions using the commands `subst(a,b,c)` and `ratsubst(a,b,c)`.

- The expression `a` will be replaced by `b` and subsequently substituted into the expression `c`.
- When using the `subst` command must be `b` the simplest part (atom) or a complete subexpression of the expression `c`.
- In the example, the subexpression is not `x+y` complete (missing `z`).
- The `ratsubst` command it also modifies the resulting expression.

(%i2) `subst(x+y,a,a^2+b^2); ratsubst(x+y,a,a^2+b^2);`

(%o1) $(y+x)^2+b^2$

(%o2) $y^2+2xy+x^2+b^2$

(%i4) `subst(a,x+y,x+y+z); ratsubst(a,x+y,x+y+z);`

(%o3) $z+y+x$

(%o4) $z+a$

Limits and Derivatives

In the `Calculus` menu we find functions for solving basic problems of mathematical analysis (limits, derivation, integration, sums of series, calculate products, . . .).

We calculate the limits using the command `limit`.

- The last parameter determines the direction of unilateral limits, has the values `plus` or `minus` and is optional.

  If not specified, Maxima calculates the limit as complex.
- With the command `limit(f(x),x,a)` we calculate the limit $\lim\limits_{x\longrightarrow a} f(x)$.
- With the command `limit(f(x),x,a,plus)` we calculate the limit $\lim\limits_{x\longrightarrow a^+} f(x)$.

```
(%i4)  limit(1/x,x,0);      limit(1/x,x,0,plus);
            limit(1/x,x,0,minus);  limit(1/x,t,0);
(%o1)  infinity
(%o2)  ∞
(%o3)  −∞
(%o4)  1
       -
       x
```

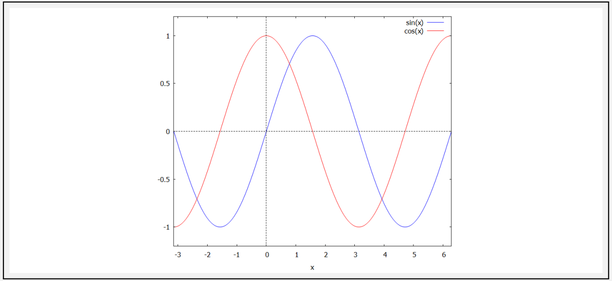
Function graphs

We can plot the function graph in several ways.

- The easiest way is to choose `Plot` in the menu submenu `Plot 2d ...`.
- If we choose `Format=gnuplot`, the function is rendered by the command `plot2d` using the Open Source program `Gnuplot` to a new window.

  `Gnuplot` is automatically installed together with Maxima.

```
(%i1) plot2d([sin(x),cos(x)],[x,-%pi,2*%pi],[y,-1.2,1.2],
          [plot_format, gnuplot])$
```

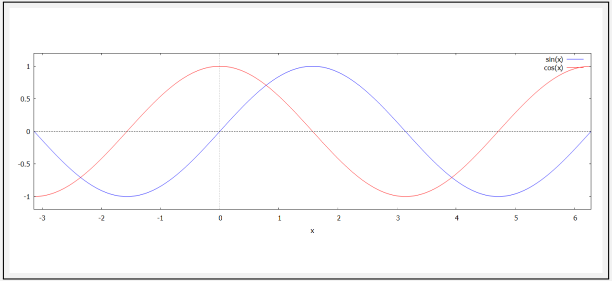


Function graphs

The graphs of functions were not displayed in the real ratio of the *x* and *y* axes, but were optimized for the screen.

- We can use e.g. `same_xy` parameter for proper display.

```
(%i1) plot2d([sin(x),cos(x)],[x,-%pi,2*%pi],[y,-1.2,1.2],
              [plot_format, gnuplot],[same_xy])$
```

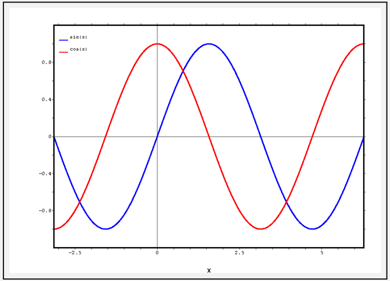


Function graphs

If we choose `Format=wxmaxima`:

- Maxima will plot the graph using the command `plot2d` to a new window.
- We can only save the image in postscript.

```
(%i1) plot2d([sin(x),cos(x)],[x,-%pi,2*%pi],[y,-1.2,1.2],
          [plot_format, xmaxima])$
```



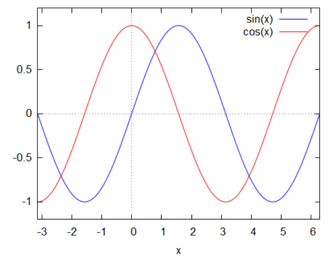
Function graphs

If we choose `Format=inline`:

- Maxima draws a graph using the command `wxplot2d` into your environment.

```
(%i1) wxplot2d([sin(x),cos(x)],[x,-%pi,2*%pi],
            [y,-1.2,1.2])$
```

(%o1)



Commands `plot2d` and `wxplot2d` they have the same syntax and have many more parameters.

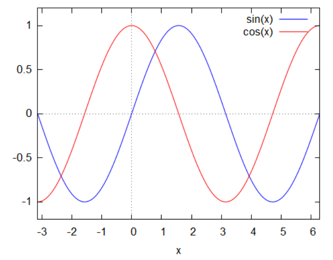
- Parameters can be found, for example, with the command `describe(plot2d)`.

Function graphs

If we choose `Format=inline`:

- Maxima draws a graph using the command `wxplot2d` into your environment.

```
(%i1) wxplot2d([sin(x),cos(x)],[x,-%pi,2*%pi],
        [y,-1.2,1.2])$
(%o1)
```



Commands `plot2d` and `wxplot2d` they have the same syntax and have many more parameters.

- Parameters can be found, for example, with the command `describe(plot2d)`.

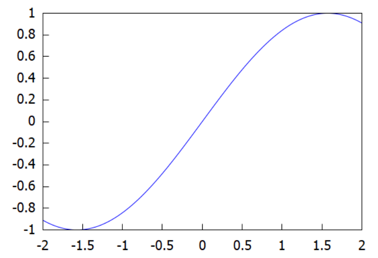
Function graphs

We can plot the function graph in several ways.

- It is better to use `wxdraw2d` or `draw2d` commands and direct the output to `Gnuplot`.
- These commands have a slightly different syntax than the `wxplot2d`, `plot2d`.
  The print parameters are simpler and clearer.
- The plotted function must be in the command `explicit`, `parametric` or `implicit`.

```
(%i1) wxdraw2d(explicit((sin(x)),x,-2,2))$
```

(%o1)



Sequences and Series

Sequences can be created in Maxima in several ways.

- Sequences can be created, for example, using the command `makelist` or with the statements of the cycle `for..do`.
- Command `makelist` creates a list that we can display as a whole and by members.

```
(%i2)  S1:makelist(2*n^2-1,n,1,10);
       S2:makelist(2*n^2-1,n,2,10,2);
(S1)   [1, 7, 17, 31, 49, 71, 97, 127, 161, 199]
(S2)   [7, 31, 71, 127, 199]
(%i4)  S1[1];S2[1];S1[10];
(%o3)  1
(%o4)  7
(%o5)  199
(%i6)  S1[12];
       inpart:  invalid index 12 of list or matrix.
       -- an error.  To debug this try:  debugmode(true);
```

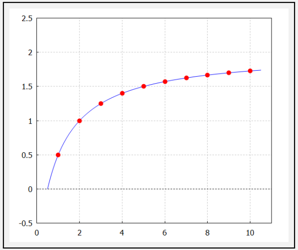
Sequences and Series

- The sequence is generated with its patterns and then plotted using the `draw2d` command.
- Arranged pairs are enclosed in square brackets and can be displayed as points in a plane.

```
(%i1) S1:makelist([n,(2*n-1)/(n+1)],n,1,10);
```

(S1) $[[1,\frac{1}{2}],[2,1],[3,\frac{5}{4}],[4,\frac{7}{5}],[5,\frac{3}{2}],[6,\frac{11}{7}],[7,\frac{13}{8}],[8,\frac{5}{3}],[9,\frac{17}{10}],[10,\frac{19}{11}]]$

```
(%i2) draw2d(grid=true,xaxis=true,yaxis=true,
      xrange=[0,11],yrange=[-0.5,2.5],
      color=blue,explicit((2*n-1)/(n+1),n,0.5,10.5),
      point_type=7,color=red,points(S1))$
```



Sequences and Series

- Using the command `for..do` we will list several members of the sequence $\{2n^2 - 1\}_{n=1}^{\infty}$.

```
(%i1)  (for n:1 thru 15 do (a_n: 2*n^2-1, print(a_n)) )$
       1
       7
       17
       31
       49
       71
       97
       127
       161
       199
       241
       287
       337
       391
       449
```

Sequences and Series

We can calculate the sum of the series with the `sum` command.
This command can be found in the menu `Calculus` and the `Calculate Sum...` submenu.

- We calculate the finite and infinite sum using the command `sum`.

```
(%i1) sum(2*n^2-1,n,1,8);
(%o1) 400
```

- Maxima can calculate the exact sum of some infinite series.

```
(%i2) sum(1/k^2,k,1,inf);

      sum(1/k^2,k,1,inf),simpsum;
```

(%o1) $\sum_{k=1}^{\infty}\left(\frac{1}{k^2}\right)$

(%o2) $\frac{\pi^2}{6}$

Real functions

Mathematical Analysis supported by wxMaxima

Basic terms

- **Binary relation** f between the sets $A \neq \emptyset$ and $B \neq \emptyset$ is every $f \subset A \times B$.
- If for each $x \in A$ there is at most one $y \in B$ such that $[x;y] \in f$, then relation f is called **function** (**map**, **mapping**, **transformation**) from set A to set B, label $f\colon A \to B$.

  We also write as $[x;y] \in f$ or $y = f(x)$.
- $x \in A$ Pattern, independent variable, input value, argument.
- $y \in B$ Image, dependent variable, output value, value of the function.
- $D(f) = \{x \in A, \exists y \in B\colon [x;y] \in f\}$ Domain of the function f (set of patterns).
- $H(f) = \{y \in B, \exists x \in D(f)\colon [x;y] \in f\}$ Codomain of values of the function f (set of images).
- Relations and functions are sets of ordered pairs.
- $f = g$ represents the equivalence of $[x;y] \in f \Leftrightarrow [x;y] \in g$,

  i.e. $D(f) = D(g)$ and $f(x) = g(x)$ for all $x \in D(f)$.

Basic terms

- **Binary relation** f between the sets $A \neq \emptyset$ and $B \neq \emptyset$ is every $f \subset A \times B$.
- If for each $x \in A$ there is at most one $y \in B$ such that $[x; y] \in f$, then relation f is called **function** (**map**, **mapping**, **transformation**) from set A to set B, label $f \colon A \to B$.

  We also write as $[x; y] \in f$ or $y = f(x)$.
- $x \in A$ Pattern, independent variable, input value, argument.
- $y \in B$ Image, dependent variable, output value, value of the function.
- $D(f) = \{x \in A, \exists y \in B \colon [x; y] \in f\}$ Domain of the function f (set of patterns).
- $H(f) = \{y \in B, \exists x \in D(f) \colon [x; y] \in f\}$ Codomain of values of the function f (set of images).
- Relations and functions are sets of ordered pairs.
- $f = g$ represents the equivalence of $[x; y] \in f \Leftrightarrow [x; y] \in g$,

  i.e. $D(f) = D(g)$ and $f(x) = g(x)$ for all $x \in D(f)$.

Basic terms

- **Binary relation** f between the sets $A \neq \emptyset$ and $B \neq \emptyset$ is every $f \subset A \times B$.
- If for each $x \in A$ there is at most one $y \in B$ such that $[x; y] \in f$, then relation f is called **function** (**map**, **mapping**, **transformation**) from set A to set B, label $f\colon A \to B$.

  We also write as $[x; y] \in f$ or $y = f(x)$.
- $x \in A$ Pattern, independent variable, input value, argument.
- $y \in B$ Image, dependent variable, output value, value of the function.
- $D(f) = \{x \in A, \exists y \in B \colon [x; y] \in f\}$ Domain of the function f (set of patterns).
- $H(f) = \{y \in B, \exists x \in D(f) \colon [x; y] \in f\}$ Codomain of values of the function f (set of images).
- Relations and functions are sets of ordered pairs.
- $f = g$ represents the equivalence of $[x; y] \in f \Leftrightarrow [x; y] \in g$,

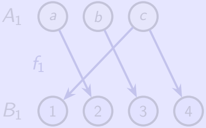
  i.e. $D(f) = D(g)$ and $f(x) = g(x)$ for all $x \in D(f)$.

Basic terms

- $\forall x_1, x_2 \in A\colon\ x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

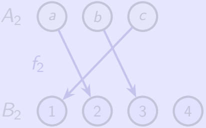
  The function f is an injection (different patterns have different images).
- $\forall y \in B\ \exists x \in A\colon\ y = f(x)$ The function f is a surjection (every image has a pattern).
- f is injection and surjection at the same time

  The function f is a bijection (injection and surjection).



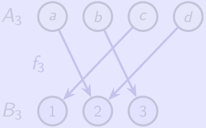
$f_1 = \{[a;2], [b;3], [c;1], [c;4]\}$

Is not a function (is a relation).



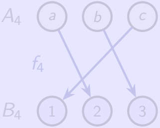
$f_2 = \{[a;2], [b;3], [c;1]\}$

Is an injection.



$f_3 = \{[a;2], [b;3], [c;1], [d;2]\}$

Is a surjection.



$f_4 = \{[a;2], [b;3], [c;1]\}$

Is a bijection.

Basic terms

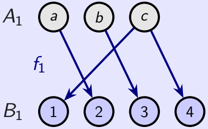
- $\forall x_1, x_2 \in A\colon\; x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

  The function f is an injection (different patterns have different images).

- $\forall y \in B\; \exists x \in A\colon\; y = f(x)$ The function f is a surjection (every image has a pattern).

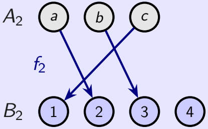
- f is injection and surjection at the same time

  The function f is a bijection (injection and surjection).



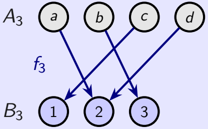
$f_1 = \{[a;2], [b;3], [c;1], [c;4]\}$

Is not a function (is a relation).



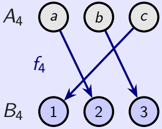
$f_2 = \{[a;2], [b;3], [c;1]\}$

Is an injection.



$f_3 = \{[a;2], [b;3], [c;1], [d;2]\}$

Is a surjection.



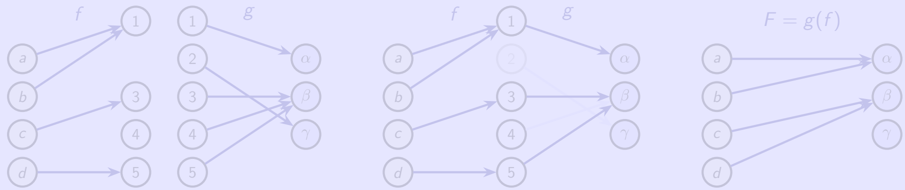
$f_4 = \{[a;2], [b;3], [c;1]\}$

Is a bijection.

Basic terms

Functions $f\colon A \to B$, $g\colon C \to D$, $H(f) \subset C$.

- The function $F = g(f)\colon A \to D$ which assigns a value to each $x \in A$, $z = g(y) = g(f(x)) \in D$, where $y = f(x)$, is called **function composition** (**composition**) of functions f and g.
- The function f is called the inner component.
- The function g is called the outer component.



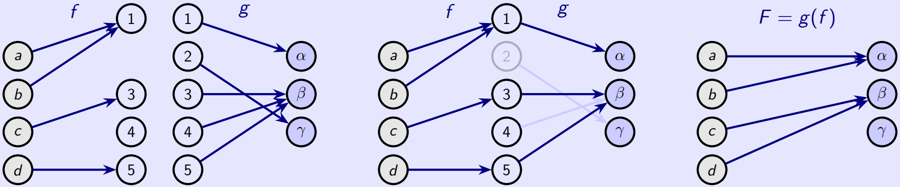
$$f = \{[a;1], [b;1], [c;3], [d;5]\},\ g = \{[1;\alpha], [2;\gamma], [3;\beta], [4;\beta], [5;\beta]\},$$

composition $F = g(f) = \{[a;\alpha], [b;\alpha], [c;\beta], [d;\beta]\}$.

Basic terms

Functions $f\colon A \to B$, $g\colon C \to D$, $H(f) \subset C$.

- The function $F = g(f)\colon A \to D$ which assigns a value to each $x \in A$, $z = g(y) = g(f(x)) \in D$, where $y = f(x)$, is called **function composition** (**composition**) of functions f and g.
- The function f is called the inner component.
- The function g is called the outer component.



$f = \{[a;1], [b;1], [c;3], [d;5]\}$, $g = \{[1;\alpha], [2;\gamma], [3;\beta], [4;\beta], [5;\beta]\}$,

composition $F = g(f) = \{[a;\alpha], [b;\alpha], [c;\beta], [d;\beta]\}$.

Basic terms

A function $f\colon A \to B$, $C \subset A$.

- The mapping $h\colon C \to B$ such that $f(x) = h(x)$ holds for all $x \in C$ is called **restriction f to the set C**, label $h = f|_C$.

A function $f\colon A \to B$ is a bijection.

- Representation of $g\colon B \to A$ such that $[y;x] \in g \Leftrightarrow [x;y] \in f$,
  i.e. $x = g(y) \Leftrightarrow y = f(x)$, is called the **inverse function to f**, label $g = f^{-1}$.

The set A is **equivalent** to the set B, if there is a bijection $f\colon A \to B$, label $A \sim B$.

$A = \emptyset$	The set A is empty.	The set A is finite.
$A \sim N_n = \{1, 2, \ldots, n\}$, $n \in N$	The set A is countable finite.	
$A \sim N$	The set A is countable infinite.	The set A is infinite.
$A \neq \emptyset$ and $A \not\sim N_n$ and $A \not\sim N$	The set A is uncountable.	
$A = \emptyset$ or $A \sim N_n$ or $A \sim N$	The set A is countable.	

Basic terms

A function $f\colon A \to B$, $C \subset A$.

- The mapping $h\colon C \to B$ such that $f(x) = h(x)$ holds for all $x \in C$ is called **restriction f to the set C**, label $h = f|_C$.

A function $f\colon A \to B$ is a bijection.

- Representation of $g\colon B \to A$ such that $[y; x] \in g \Leftrightarrow [x; y] \in f$, i.e. $x = g(y) \Leftrightarrow y = f(x)$, is called the **inverse function to f**, label $g = f^{-1}$.

The set A is **equivalent** to the set B, if there is a bijection $f\colon A \to B$, label $A \sim B$.

$A = \emptyset$	The set A is empty.	The set A is finite.
$A \sim N_n = \{1, 2, \dots, n\}$, $n \in N$	The set A is countable finite.	
$A \sim N$	The set A is countable infinite.	The set A is infinite.
$A \neq \emptyset$ and $A \not\sim N_n$ and $A \not\sim N$	The set A is uncountable.	
$A = \emptyset$ or $A \sim N_n$ or $A \sim N$	The set A is countable.	

Basic terms

A function $f\colon A \to B$, $C \subset A$.

- The mapping $h\colon C \to B$ such that $f(x) = h(x)$ holds for all $x \in C$ is called **restriction f to the set** C, label $h = f|_C$.

A function $f\colon A \to B$ is a bijection.

- Representation of $g\colon B \to A$ such that $[y;x] \in g \Leftrightarrow [x;y] \in f$, i.e. $x = g(y) \Leftrightarrow y = f(x)$, is called the **inverse function to** f, label $g = f^{-1}$.

The set A is **equivalent** to the set B, if there is a bijection $f\colon A \to B$, label $A \sim B$.

$A = \emptyset$	The set A is empty.	The set A is finite.
$A \sim N_n = \{1, 2, \ldots, n\}$, $n \in N$	The set A is countable finite.	
$A \sim N$	The set A is countable infinite.	The set A is infinite.
$A \neq \emptyset$ and $A \not\sim N_n$ and $A \not\sim N$	The set A is uncountable.	
$A = \emptyset$ or $A \sim N_n$ or $A \sim N$	The set A is countable.	

Basic terms

- $N = \{1, 2, 3, \ldots, n, n{+}1, \ldots\}$ Natural numbers.
- $Z = \{m - n, m, n \in N\} = \{0, \pm 1, \pm 2, \ldots, \pm n, \ldots\}$ Integer numbers.
- $Q = \{\frac{m}{n}, m \in Z, n \in N\}$ Rational numbers.

  The sum, difference, product and quotient of two rational numbers (with a non-zero denominator) is again a rational number.
  A rational number (fraction) can have several different expressions.
- $I = R - Q$ Irrational numbers.

  The sum, difference, product and quotient of two irrational numbers it can be irrational as well as rational.
- $R = (-\infty; \infty)$ Real numbers.

  The set R is infinite, but all its elements, i.e. numbers are finite (the number of elements of the set R cannot be expressed by a number).
- $R^* = R \cup \{-\infty, \infty\}$ Extended set of real numbers.
- $\infty + \infty = \infty$, $a \pm \infty = \pm\infty$, $\infty \cdot \infty = \infty$, $b \cdot \infty = \frac{\infty}{b} = \infty$, $\frac{a}{\infty} = 0$ for $a, b \in R$, $b > 0$.
- We do not define $\infty - \infty$, $\infty \cdot 0$, $\frac{\infty}{\infty}$, $\frac{\infty}{0}$, $\frac{a}{0}$ for $a \in R$ (indefinite expressions).

Basic terms

- $N = \{1, 2, 3, \ldots, n, n+1, \ldots\}$ Natural numbers.
- $Z = \{m - n, m, n \in N\} = \{0, \pm 1, \pm 2, \ldots, \pm n, \ldots\}$ Integer numbers.
- $Q = \{\frac{m}{n}, m \in Z, n \in N\}$ Rational numbers.

  The sum, difference, product and quotient of two rational numbers (with a non-zero denominator) is again a rational number.
  A rational number (fraction) can have several different expressions.
- $I = R - Q$ Irrational numbers.

  The sum, difference, product and quotient of two irrational numbers it can be irrational as well as rational.
- $R = (-\infty; \infty)$ Real numbers.

  The set R is infinite, but all its elements, i.e. numbers are finite (the number of elements of the set R cannot be expressed by a number).
- $R^* = R \cup \{-\infty, \infty\}$ Extended set of real numbers.
- $\infty + \infty = \infty$, $a \pm \infty = \pm\infty$, $\infty \cdot \infty = \infty$, $b \cdot \infty = \frac{\infty}{b} = \infty$, $\frac{a}{\infty} = 0$ for $a, b \in R$, $b > 0$.
- We do not define $\infty - \infty$, $\infty \cdot 0$, $\frac{\infty}{\infty}$, $\frac{\infty}{0}$, $\frac{a}{0}$ for $a \in R$ (indefinite expressions).

Basic terms

- $N=\{1,2,3,\ldots,n,n{+}1,\ldots\}$ Natural numbers.
- $Z=\{m-n, m,n\in N\}=\{0,\pm1,\pm2,\ldots,\pm n,\ldots\}$ Integer numbers.
- $Q=\{\frac{m}{n}, m\in Z, n\in N\}$ Rational numbers.
  The sum, difference, product and quotient of two rational numbers (with a non-zero denominator) is again a rational number.
  A rational number (fraction) can have several different expressions.
- $I=R-Q$ Irrational numbers.
  The sum, difference, product and quotient of two irrational numbers it can be irrational as well as rational.
- $R=(-\infty;\infty)$ Real numbers.
  The set R is infinite, but all its elements, i.e. numbers are finite (the number of elements of the set R cannot be expressed by a number).
- $R^*=R\cup\{-\infty,\infty\}$ Extended set of real numbers.
- $\infty+\infty=\infty$, $a\pm\infty=\pm\infty$, $\infty\cdot\infty=\infty$, $b\cdot\infty=\frac{\infty}{b}=\infty$, $\frac{a}{\infty}=0$ for $a,b\in R$, $b>0$.
- We do not define $\infty-\infty$, $\infty\cdot 0$, $\frac{\infty}{\infty}$, $\frac{\infty}{0}$, $\frac{a}{0}$ for $a\in R$ (indefinite expressions).

Sequences (of real numbers)

$$\left.\begin{array}{l}\text{A function } f,\ D(f) = N\\ f = \{[n; f(n)]\,, n\in N\}\end{array}\right\}$$ **Sequence**, for $n\in N$ denote $a_n = f(n)$,

i.e. $f = \{a_1, a_2, a_3, \ldots, a_n, \ldots\} = \{a_n\}_{n=1}^{\infty}$.

- $f \sim N$ The sequence f is countable (infinite).
- $a_n \in f$ A member of the sequence, represents $[n; f(n)]$,

  i.e. pattern (order n) and image $a_n = f(n)$.

Sequence (of real numbers) is every sequence $\{a_n\}_{n=1}^{\infty}$ where $a_n\in R$, i.e. $f : N \to R$, $D(f) \in R$.

- Explicit entry: General expression $a_n = f(n)$, $n\in N$.

  $a_n = n^2$, $n\in N$ defines the sequence $\{a_n\}_{n=1}^{\infty} = \left\{n^2\right\}_{n=1}^{\infty} = \{1, 4, 9, 16, \ldots\}$.

- Recurring entry: Given a_1 and given a_n, $n\in N$ using the previous members.

  $a_1 = 1$, $a_{n+1} = a_n + 2n + 1$, $n\in N$

  defines the sequence $\{a_n\}_{n=1}^{\infty} = \left\{n^2\right\}_{n=1}^{\infty} = \{1, 4, 9, 16, \ldots\}$.

Sequences (of real numbers)

$$\left.\begin{array}{l}\text{A function } f,\ D(f)=N \\ f=\{[n;f(n)]\,,n\in N\}\end{array}\right\}$$ **Sequence**, for $n\in N$ denote $a_n=f(n)$,
i.e. $f=\{a_1,a_2,a_3,\dots,a_n,\dots\}=\{a_n\}_{n=1}^{\infty}$.

- $f\sim N$ The sequence f is countable (infinite).
- $a_n\in f$ A member of the sequence, represents $[n;f(n)]$,
  i.e. pattern (order n) and image $a_n=f(n)$.

Sequence (of real numbers) is every sequence $\{a_n\}_{n=1}^{\infty}$ where $a_n\in R$, i.e. $f\colon N\to R$, $D(f)\in R$.

- Explicit entry: General expression $a_n=f(n)$, $n\in N$.
  $a_n=n^2$, $n\in N$ defines the sequence $\{a_n\}_{n=1}^{\infty}=\{n^2\}_{n=1}^{\infty}=\{1,4,9,16,\dots\}$.
- Recurring entry: Given a_1 and given a_n, $n\in N$ using the previous members.
  $a_1=1$, $a_{n+1}=a_n+2n+1$, $n\in N$
  defines the sequence $\{a_n\}_{n=1}^{\infty}=\{n^2\}_{n=1}^{\infty}=\{1,4,9,16,\dots\}$.

Sequences (of real numbers)

$$\left.\begin{array}{l}\text{A function } f,\ D(f)=N\\ f=\{[n;f(n)]\,,n\in N\}\end{array}\right\}\ \textbf{Sequence}\text{, for } n\in N \text{ denote } a_n=f(n),$$

$$\text{i.e. } f=\{a_1,a_2,a_3,\ldots,a_n,\ldots\}=\{a_n\}_{n=1}^{\infty}.$$

- $f\sim N$ The sequence f is countable (infinite).
- $a_n\in f$ A member of the sequence, represents $[n;f(n)]$, i.e. pattern (order n) and image $a_n=f(n)$.

Sequence (of real numbers) is every sequence $\{a_n\}_{n=1}^{\infty}$ where $a_n\in R$, i.e. $f\colon N\to R$, $D(f)\in R$.

- Explicit entry: General expression $a_n=f(n)$, $n\in N$.

  $a_n=n^2$, $n\in N$ defines the sequence $\{a_n\}_{n=1}^{\infty}=\left\{n^2\right\}_{n=1}^{\infty}=\{1,4,9,16,\ldots\}$.

- Recurring entry: Given a_1 and given a_n, $n\in N$ using the previous members.

  $a_1=1$, $a_{n+1}=a_n+2n+1$, $n\in N$

  defines the sequence $\{a_n\}_{n=1}^{\infty}=\left\{n^2\right\}_{n=1}^{\infty}=\{1,4,9,16,\ldots\}$.

Sequences (of real numbers)

A sequence $\{a_n\}_{n=1}^{\infty}$, $a_n \in R$, numbers $a, b \in R$.

- $\forall n \in N\colon\ a \le a_n$ a is the lower bound, $\{a_n\}_{n=1}^{\infty}$ is bounded from below.
- $\forall n \in N\colon\ a_n \le b$ b is an upper bound, $\{a_n\}_{n=1}^{\infty}$ is bounded from above.
- $\{a_n\}_{n=1}^{\infty}$ is bounded from below and above $\{a_n\}_{n=1}^{\infty}$ is bounded.

$$\left.\begin{array}{l} \left.\begin{array}{rl} \forall n \in N\colon\ a_n < a_{n+1} & \text{Increasing.} \\ a_n > a_{n+1} & \text{Decreasing.} \end{array}\right\} \text{Strictly monotonic.} \\ \begin{array}{rl} \forall n \in N\colon\ a_n \le a_{n+1} & \text{Non-decreasing.} \\ a_n \ge a_{n+1} & \text{Non-increasing.} \\ a_n = a_{n+1} & \text{Constant (stationary).} \end{array} \end{array}\right\} \text{Monotonic sequence } \{a_n\}_{n=1}^{\infty}.$$

- The sequence $\{a_n\}_{n=1}^{\infty} = \{3, 1, 3, 5, 7, 9, \ldots\}$ is not monotonic.

Sequences (of real numbers)

A sequence $\{a_n\}_{n=1}^{\infty}$, $a_n \in R$, numbers $a, b \in R$.

- $\forall n \in N\colon\ a \leq a_n$ a is the lower bound, $\{a_n\}_{n=1}^{\infty}$ is bounded from below.
- $\forall n \in N\colon\ a_n \leq b$ b is an upper bound, $\{a_n\}_{n=1}^{\infty}$ is bounded from above.
- $\{a_n\}_{n=1}^{\infty}$ is bounded from below and above $\{a_n\}_{n=1}^{\infty}$ is bounded.

$$\left.\begin{array}{l} \left.\begin{array}{lll} \forall n\in N\colon & a_n < a_{n+1} & \text{Increasing.}\\ & a_n > a_{n+1} & \text{Decreasing.}\end{array}\right\}\text{Strictly monotonic.}\\ \\ \begin{array}{lll} \forall n\in N\colon & a_n \leq a_{n+1} & \text{Non-decreasing.}\\ & a_n \geq a_{n+1} & \text{Non-increasing.}\\ & a_n = a_{n+1} & \text{Constant (stationary).}\end{array}\end{array}\right\}\text{Monotonic sequence } \{a_n\}_{n=1}^{\infty}.$$

- The sequence $\{a_n\}_{n=1}^{\infty} = \{3, 1, 3, 5, 7, 9, \ldots\}$ is not monotonic.

Sequences (of real numbers)

A sequence $\{a_n\}_{n=1}^{\infty}$, $a_n \in R$, numbers $a, b \in R$.

- $\forall n \in N$: $a \le a_n$ a is the lower bound, $\{a_n\}_{n=1}^{\infty}$ is bounded from below.
- $\forall n \in N$: $a_n \le b$ b is an upper bound, $\{a_n\}_{n=1}^{\infty}$ is bounded from above.
- $\{a_n\}_{n=1}^{\infty}$ is bounded from below and above $\{a_n\}_{n=1}^{\infty}$ is bounded.

$$\left.\begin{array}{l} \left.\begin{array}{lll} \forall n\in N: & a_n < a_{n+1} & \text{Increasing.}\\ & a_n > a_{n+1} & \text{Decreasing.}\end{array}\right\} \text{Strictly monotonic.} \\ \begin{array}{lll} \forall n\in N: & a_n \le a_{n+1} & \text{Non-decreasing.}\\ & a_n \ge a_{n+1} & \text{Non-increasing.}\\ & a_n = a_{n+1} & \text{Constant (stationary).}\end{array}\end{array}\right\} \text{Monotonic sequence } \{a_n\}_{n=1}^{\infty}.$$

- The sequence $\{a_n\}_{n=1}^{\infty} = \{3, 1, 3, 5, 7, 9, \ldots\}$ is not monotonic.

Sequences (of real numbers)

A sequence $\{a_n\}_{n=1}^{\infty}$, $a_n \in R$.

- If $\{k_n\}_{n=1}^{\infty} \subset N$ is an increasing sequence (of natural numbers, indices), then $\{a_{k_n}\}_{n=1}^{\infty}$ is called **subsequence** (**selected sequence**) from $\{a_n\}_{n=1}^{\infty}$.

Subsequences of the $\{a_n\}_{n=1}^{\infty} = \{2n-1\}_{n=1}^{\infty} = \{1, 3, 5, 7, 9, 11, 13, \ldots\}$ are for example:

- $\{a_{k_n}\}_{n=1}^{\infty} = \{a_{2n}\}_{n=1}^{\infty} = \{a_2, a_4, a_6, \ldots\} = \{3, 7, 11, \ldots\} = \{4n-1\}_{n=1}^{\infty}$.
- $\{a_n\}_{n=1}^{\infty} = \{2n-1\}_{n=1}^{\infty}$.
- $\{a_n\}_{n=2}^{\infty} = \{2n-1\}_{n=2}^{\infty}$.
- $\{101, 109, 235, 637, \ldots\}$.

```
(%i2)  a(n):=2*n-1$ makelist(a(n),n,1,7);
(%o2)  [1,3,5,7,9,11,13]
(%i3)  makelist(a(2*n),n,1,7);
(%o3)  [3,7,11,15,19,23,27]
(%i4)  makelist(a(2*n),n,2,7);
(%o4)  [7,11,15,19,23,27]
(%i5)  print(a(51),a(55),a(118),a(319))$
       101 109 235 637
```

Sequences (of real numbers)

A sequence $\{a_n\}_{n=1}^{\infty}$, $a_n \in R$.

- If $\{k_n\}_{n=1}^{\infty} \subset N$ is an increasing sequence (of natural numbers, indices), then $\{a_{k_n}\}_{n=1}^{\infty}$ is called **subsequence** (**selected sequence**) from $\{a_n\}_{n=1}^{\infty}$.

Subsequences of the $\{a_n\}_{n=1}^{\infty} = \{2n-1\}_{n=1}^{\infty} = \{1, 3, 5, 7, 9, 11, 13, \ldots\}$ are for example:

- $\{a_{k_n}\}_{n=1}^{\infty} = \{a_{2n}\}_{n=1}^{\infty} = \{a_2, a_4, a_6, \ldots\} = \{3, 7, 11, \ldots\} = \{4n-1\}_{n=1}^{\infty}$.
- $\{a_n\}_{n=1}^{\infty} = \{2n-1\}_{n=1}^{\infty}$.
- $\{a_n\}_{n=2}^{\infty} = \{2n-1\}_{n=2}^{\infty}$.
- $\{101, 109, 235, 637, \ldots\}$.

```
(%i2)  a(n):=2*n-1$ makelist(a(n),n,1,7);
(%o2)  [1, 3, 5, 7, 9, 11, 13]
(%i3)  makelist(a(2*n),n,1,7);
(%o3)  [3, 7, 11, 15, 19, 23, 27]
(%i4)  makelist(a(2*n),n,2,7);
(%o4)  [7, 11, 15, 19, 23, 27]
(%i5)  print(a(51),a(55),a(118),a(319))$
       101  109  235  637
```

Sequences (of real numbers)

For every neighborhood $O(a)$ there are infinitely many members $a_n \in O(a)$, then $a \in R^* = R \cup \{\pm\infty\}$ is called **accumulation value** of the sequence $\{a_n\}_{n=1}^{\infty}$.

- The set of all accumulation values of the sequence $\{a_n\}_{n=1}^{\infty}$ we denote by E.

$\sup E = \limsup\limits_{n\to\infty} a_n$ **Limes superior** (upper limit) of the sequence $\{a_n\}_{n=1}^{\infty}$.

$\inf E = \liminf\limits_{n\to\infty} a_n$ **Limes inferior** (lower limit) of the sequence $\{a_n\}_{n=1}^{\infty}$.

They always exist.

$\sup E = \inf E = \lim\limits_{n\to\infty} a_n$ **Limit** $\{a_n\}_{n=1}^{\infty}$ (set E has a single element).

A sequence $\{a_n\}_{n=1}^{\infty}$.

- The sequence $\{a_n\}_{n=1}^{\infty}$ has at least one accumulation value.
- If $\lim\limits_{n\to\infty} a_n$ exists, then it is unique.

Sequences (of real numbers)

For every neighborhood $O(a)$ there are infinitely many members $a_n \in O(a)$,
then $a \in R^* = R \cup \{\pm\infty\}$ is called **accumulation value** of the sequence $\{a_n\}_{n=1}^{\infty}$.

- The set of all accumulation values of the sequence $\{a_n\}_{n=1}^{\infty}$ we denote by E.

$\sup E = \limsup\limits_{n\to\infty} a_n$ **Limes superior** (upper limit) of the sequence $\{a_n\}_{n=1}^{\infty}$.

$\inf E = \liminf\limits_{n\to\infty} a_n$ **Limes inferior** (lower limit) of the sequence $\{a_n\}_{n=1}^{\infty}$.

They always exist.

$\sup E = \inf E = \lim\limits_{n\to\infty} a_n$ **Limit** $\{a_n\}_{n=1}^{\infty}$ (set E has a single element).

A sequence $\{a_n\}_{n=1}^{\infty}$.

- The sequence $\{a_n\}_{n=1}^{\infty}$ has at least one accumulation value.
- If $\lim\limits_{n\to\infty} a_n$ exists, then it is unique.

Sequences (of real numbers)

$\exists \lim\limits_{n\to\infty} a_n = a \in R$

Exists a finite limit,

$\{a_n\}_{n=1}^{\infty}$ converges to the number a,

$\{a_n\}_{n=1}^{\infty} \longrightarrow a.$

$\{a_n\}_{n=1}^{\infty}$ converges,

$\{a_n\}_{n=1}^{\infty} \longrightarrow.$

$\exists \lim\limits_{n\to\infty} a_n = \pm\infty$

Exists an infinite limit,

$\{a_n\}_{n=1}^{\infty}$ diverges to $\pm\infty$,

$\{a_n\}_{n=1}^{\infty} \longrightarrow \pm\infty.$

$\nexists \lim\limits_{n\to\infty} a_n$

Do not exists limit,

$\{a_n\}_{n=1}^{\infty}$ oscillates.

$\{a_n\}_{n=1}^{\infty}$ diverges,

$\{a_n\}_{n=1}^{\infty} \not\longrightarrow.$

A sequence $\{a_n\}_{n=1}^{\infty}$.

- Changing the finite number (replacement, omission, addition, etc.) of members of the sequence $\{a_n\}_{n=1}^{\infty}$ does not affect the convergence, or divergence of this sequence.

Sequences (of real numbers)

A sequence $\{a_n\}_{n=1}^{\infty}$.

- $\{a_n\}_{n=1}^{\infty} \longrightarrow$. $\Rightarrow$ • $\{a_n\}_{n=1}^{\infty}$ is bounded.
- $\{a_n\}_{n=1}^{\infty}$ is monotonic. $\Rightarrow$ • $\{a_n\}_{n=1}^{\infty} \longrightarrow a \in R^*$.

$$L=\lim_{n\to\infty} n^q=\begin{cases} \infty^q=\infty. & \Rightarrow \bullet\ L=\infty \text{ for } q>0.\\ \lim\limits_{n\to\infty} n^0=\lim\limits_{n\to\infty} 1=1. & \Rightarrow \bullet\ L=1 \text{ for } q=0.\\ \lim\limits_{n\to\infty}\frac{1}{n^{-q}}=\frac{1}{\infty}=0. & \Rightarrow \bullet\ L=0 \text{ for } q<0\ (-q>0).\end{cases}$$

Geometric sequence.

$$L=\lim_{n\to\infty} q^n \begin{cases} q^n\to\infty. & \Rightarrow \bullet\ L=\infty \text{ for } q>1.\\ q^n=1^n=1. & \Rightarrow \bullet\ L=1 \text{ for } q=1.\\ q^n\to 0. & \Rightarrow \bullet\ L=0 \text{ for } q\in(0;1).\\ q^n=q^{2k}=1,\ q^n=q^{2k+1}=-1. & \Rightarrow \bullet\ \nexists \text{ for } q=-1.\\ q^n=q^{2k}\to\infty,\ q^n=q^{2k+1}\to-\infty. & \Rightarrow \bullet\ \nexists \text{ for } q<-1.\end{cases}$$

Sequences (of real numbers)

A sequence $\{a_n\}_{n=1}^{\infty}$.

- $\{a_n\}_{n=1}^{\infty} \longrightarrow$. $\Rightarrow$ • $\{a_n\}_{n=1}^{\infty}$ is bounded.
- $\{a_n\}_{n=1}^{\infty}$ is monotonic. $\Rightarrow$ • $\{a_n\}_{n=1}^{\infty} \longrightarrow a \in R^*$.

$$L = \lim_{n\to\infty} n^q = \begin{cases} \infty^q = \infty. & \Rightarrow \bullet\ L = \infty \text{ for } q > 0. \\ \lim\limits_{n\to\infty} n^0 = \lim\limits_{n\to\infty} 1 = 1. & \Rightarrow \bullet\ L = 1 \text{ for } q = 0. \\ \lim\limits_{n\to\infty} \frac{1}{n^{-q}} = \frac{1}{\infty} = 0. & \Rightarrow \bullet\ L = 0 \text{ for } q < 0\ (-q > 0). \end{cases}$$

Geometric sequence.

$$L = \lim_{n\to\infty} q^n \begin{cases} q^n \to \infty. & \Rightarrow \bullet\ L = \infty \text{ for } q > 1. \\ q^n = 1^n = 1. & \Rightarrow \bullet\ L = 1 \text{ for } q = 1. \\ q^n \to 0. & \Rightarrow \bullet\ L = 0 \text{ for } q \in (0; 1). \\ q^n = q^{2k} = 1,\ q^n = q^{2k+1} = -1. & \Rightarrow \bullet\ \nexists \text{ for } q = -1. \\ q^n = q^{2k} \to \infty,\ q^n = q^{2k+1} \to -\infty. & \Rightarrow \bullet\ \nexists \text{ for } q < -1. \end{cases}$$

Sequences (of real numbers)

A sequence $\{a_n\}_{n=1}^{\infty}$.

- $\{a_n\}_{n=1}^{\infty} \longrightarrow .$ $\quad\Rightarrow\quad$ • $\{a_n\}_{n=1}^{\infty}$ is bounded.
- $\{a_n\}_{n=1}^{\infty}$ is monotonic. $\quad\Rightarrow\quad$ • $\{a_n\}_{n=1}^{\infty} \longrightarrow a \in R^*.$

$$L = \lim_{n\to\infty} n^q = \begin{cases} \infty^q = \infty. & \Rightarrow \ \bullet \ L = \infty \ \text{for } q > 0. \\ \lim\limits_{n\to\infty} n^0 = \lim\limits_{n\to\infty} 1 = 1. & \Rightarrow \ \bullet \ L = 1 \ \text{ for } q = 0. \\ \lim\limits_{n\to\infty} \frac{1}{n^{-q}} = \frac{1}{\infty} = 0. & \Rightarrow \ \bullet \ L = 0 \ \text{ for } q < 0 \ (-q > 0). \end{cases}$$

Geometric sequence.

$$L = \lim_{n\to\infty} q^n \begin{cases} q^n \to \infty. & \Rightarrow \ \bullet \ L = \infty \ \text{for } q > 1. \\ q^n = 1^n = 1. & \Rightarrow \ \bullet \ L = 1 \ \text{ for } q = 1. \\ q^n \to 0. & \Rightarrow \ \bullet \ L = 0 \ \text{ for } q \in (0;1). \\ q^n = q^{2k} = 1,\ q^n = q^{2k+1} = -1. & \Rightarrow \ \bullet \ \nexists \ \text{ for } q = -1. \\ q^n = q^{2k} \to \infty,\ q^n = q^{2k+1} \to -\infty. & \Rightarrow \ \bullet \ \nexists \ \text{ for } q < -1. \end{cases}$$

Sequences (of real numbers)

A sequence $\{a_n\}_{n=1}^{\infty}$.

- $a_n > 0,\ n \in N.$ $\Rightarrow$ • $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n}$ (if limits exist).

- $a_n \geq 0,\ n \in N,\ \lim\limits_{n\to\infty} \sqrt[n]{a_n} = a \in R^*.$ $\Rightarrow$ • $\lim\limits_{n\to\infty} a_n = \begin{cases} 0 & \text{for } a < 1, \\ \infty & \text{for } a > 1. \end{cases}$
- $a_n > 0,\ n \in N,\ \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = a \in R^*.$ $\Rightarrow$ • $\lim\limits_{n\to\infty} a_n = \begin{cases} 0 & \text{for } a < 1, \\ \infty & \text{for } a > 1. \end{cases}$

Important limits. $a, b \in R,\ a > 0.$

- $\lim\limits_{n\to\infty} \sqrt[n]{n!} = \infty.$
- $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1.$
- $\lim\limits_{n\to\infty} \sqrt[n]{a} = 1.$
- $\lim\limits_{n\to\infty} \frac{n^n}{n!} = \infty.$
- $\lim\limits_{n\to\infty} \left(1 + \frac{b}{n}\right)^n = e^b.$
- $\lim\limits_{n\to\infty} n\left(\sqrt[n]{a} - 1\right) = \ln a.$
- $\lim\limits_{n\to\infty} \frac{n!}{n^n} = 0.$
- $\lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.$
- $\lim\limits_{n\to\infty} n\left(\sqrt[n]{e} - 1\right) = \ln e = 1.$

- The number e is called **Euler's number**. Its value is approximately 2,718 281 827.

Sequences (of real numbers)

A sequence $\{a_n\}_{n=1}^{\infty}$.

- $a_n > 0,\ n \in N.$ $\Rightarrow$ • $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n}$ (if limits exist).

- $a_n \geq 0,\ n \in N,\ \lim\limits_{n\to\infty} \sqrt[n]{a_n} = a \in R^*.$ $\Rightarrow$ • $\lim\limits_{n\to\infty} a_n = \begin{cases} 0 & \text{for } a < 1, \\ \infty & \text{for } a > 1. \end{cases}$
- $a_n > 0,\ n \in N,\ \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = a \in R^*.$ $\Rightarrow$ • $\lim\limits_{n\to\infty} a_n = \begin{cases} 0 & \text{for } a < 1, \\ \infty & \text{for } a > 1. \end{cases}$

Important limits. $a, b \in R,\ a > 0.$

- $\lim\limits_{n\to\infty} \sqrt[n]{n!} = \infty.$
- $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1.$
- $\lim\limits_{n\to\infty} \sqrt[n]{a} = 1.$
- $\lim\limits_{n\to\infty} \frac{n^n}{n!} = \infty.$
- $\lim\limits_{n\to\infty} \left(1 + \frac{b}{n}\right)^n = e^b.$
- $\lim\limits_{n\to\infty} n\left(\sqrt[n]{a} - 1\right) = \ln a.$
- $\lim\limits_{n\to\infty} \frac{n!}{n^n} = 0.$
- $\lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.$
- $\lim\limits_{n\to\infty} n\left(\sqrt[n]{e} - 1\right) = \ln e = 1.$

- The number e is called **Euler's number**. Its value is approximately $2,718\,281\,827$.

Sequences (of real numbers)

A sequence $\{a_n\}_{n=1}^{\infty}$.

- $a_n > 0,\ n \in N.$ $\Rightarrow$ • $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n}$ (if limits exist).

- $a_n \geq 0,\ n \in N,\ \lim\limits_{n\to\infty} \sqrt[n]{a_n} = a \in R^*.$ $\Rightarrow$ • $\lim\limits_{n\to\infty} a_n = \begin{cases} 0 & \text{for } a < 1, \\ \infty & \text{for } a > 1. \end{cases}$
- $a_n > 0,\ n \in N,\ \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = a \in R^*.$ $\Rightarrow$ • $\lim\limits_{n\to\infty} a_n = \begin{cases} 0 & \text{for } a < 1, \\ \infty & \text{for } a > 1. \end{cases}$

Important limits. $a, b \in R,\ a > 0.$

- $\lim\limits_{n\to\infty} \sqrt[n]{n!} = \infty.$
- $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1.$
- $\lim\limits_{n\to\infty} \sqrt[n]{a} = 1.$
- $\lim\limits_{n\to\infty} \frac{n^n}{n!} = \infty.$
- $\lim\limits_{n\to\infty} \left(1 + \frac{b}{n}\right)^n = \mathrm{e}^b.$
- $\lim\limits_{n\to\infty} n\left(\sqrt[n]{a} - 1\right) = \ln a.$
- $\lim\limits_{n\to\infty} \frac{n!}{n^n} = 0.$
- $\lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \mathrm{e}.$
- $\lim\limits_{n\to\infty} n\left(\sqrt[n]{\mathrm{e}} - 1\right) = \ln \mathrm{e} = 1.$

- The number e is called **Euler's number**. Its value is approximately 2, 718 281 827.

Number series

If $\{a_n\}_{n=1}^{\infty}$ is a sequence,

then $\sum\limits_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots$ is called **(infinite number) series**.

- Number series are closely related to sequences and generalize the concept additions to an infinite number of addends. A simple example is fractions and periodic numbers.

$$\sum_{n=1}^{\infty} a_n = \underbrace{a_1 + a_2 + a_3 + \cdots + a_k}_{s_k=\sum_{i=1}^{k} a_i \ (k\text{-th partial sum})} + \underbrace{a_{k+1} + a_{k+2} + a_{k+3} + \cdots}_{r_k=\sum_{i=k+1}^{\infty} a_i \ (k\text{-th rest})}$$

- $\{s_k\}_{k=1}^{\infty} = \{s_1, s_2, s_3, \dots\} = \{s_n\}_{n=1}^{\infty}$ The sequence of partial sums of the series $\sum\limits_{n=1}^{\infty} a_n$.
- The relationship between $\sum\limits_{n=1}^{\infty} a_n$ and the sequence $\{s_n\}_{n=1}^{\infty}$ is mutually unambiguous.
    - $s_1 = a_1 = s_0 + a_1$.
    - $s_2 = a_1 + a_2 = s_1 + a_2$.
    - $s_3 = a_1 + a_2 + a_3 = s_2 + a_3$.

      $\cdots$
    - $s_n = a_1 + a_2 + \cdots + a_{n-1} + a_n = s_{n-1} + a_n$.
    - $a_1 = s_1 - s_0$, where $s_0 = 0$.
    - $a_2 = s_2 - s_1$.
    - $a_3 = s_3 - s_2$.
    - $a_n = s_n - s_{n-1}$, $n \in N$.

Number series

If $\{a_n\}_{n=1}^{\infty}$ is a sequence,

then $\sum\limits_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots$ is called **(infinite number) series**.

- Number series are closely related to sequences and generalize the concept additions to an infinite number of addends. A simple example is fractions and periodic numbers.

$$\sum_{n=1}^{\infty} a_n = \underbrace{a_1 + a_2 + a_3 + \cdots + a_k}_{s_k = \sum_{i=1}^{k} a_i \ (k\text{-th partial sum})} + \underbrace{a_{k+1} + a_{k+2} + a_{k+3} + \cdots}_{r_k = \sum_{i=k+1}^{\infty} a_i \ (k\text{-th rest})}$$

- $\{s_k\}_{k=1}^{\infty} = \{s_1, s_2, s_3, \ldots\} = \{s_n\}_{n=1}^{\infty}$ The sequence of partial sums of the series $\sum\limits_{n=1}^{\infty} a_n$.
- The relationship between $\sum\limits_{n=1}^{\infty} a_n$ and the sequence $\{s_n\}_{n=1}^{\infty}$ is mutually unambiguous.
  - $s_1 = a_1 = s_0 + a_1$.
  - $s_2 = a_1 + a_2 = s_1 + a_2$.
  - $s_3 = a_1 + a_2 + a_3 = s_2 + a_3$.

    $\cdots$
  - $s_n = a_1 + a_2 + \cdots + a_{n-1} + a_n = s_{n-1} + a_n$.
  - $a_1 = s_1 - s_0$, where $s_0 = 0$.
  - $a_2 = s_2 - s_1$.
  - $a_3 = s_3 - s_2$.
  - $a_n = s_n - s_{n-1}$, $n \in N$.

Number series

If $\{a_n\}_{n=1}^{\infty}$ is a sequence,

then $\sum\limits_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots$ is called **(infinite number) series**.

- Number series are closely related to sequences and generalize the concept additions to an infinite number of addends. A simple example is fractions and periodic numbers.

$$\sum_{n=1}^{\infty} a_n = \underbrace{a_1 + a_2 + a_3 + \cdots + a_k}_{s_k=\sum\limits_{i=1}^{k} a_i \ (k\text{-th partial sum})} + \underbrace{a_{k+1} + a_{k+2} + a_{k+3} + \cdots}_{r_k=\sum\limits_{i=k+1}^{\infty} a_i \ (k\text{-th rest})}$$

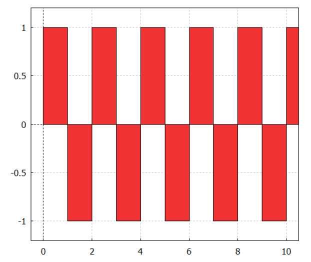
- $\{s_k\}_{k=1}^{\infty} = \{s_1, s_2, s_3, \ldots\} = \{s_n\}_{n=1}^{\infty}$ The sequence of partial sums of the series $\sum\limits_{n=1}^{\infty} a_n$.
- The relationship between $\sum\limits_{n=1}^{\infty} a_n$ and the sequence $\{s_n\}_{n=1}^{\infty}$ is mutually unambiguous.
  - $s_1 = a_1 = s_0 + a_1$.
  - $s_2 = a_1 + a_2 = s_1 + a_2$.
  - $s_3 = a_1 + a_2 + a_3 = s_2 + a_3$.

    ...
  - $s_n = a_1 + a_2 + \cdots + a_{n-1} + a_n = s_{n-1} + a_n$.
  - $a_1 = s_1 - s_0$, where $s_0 = 0$.
  - $a_2 = s_2 - s_1$.
  - $a_3 = s_3 - s_2$.
  - $a_n = s_n - s_{n-1}$, $n \in N$.

Number series

Series $\sum_{n=1}^{\infty}(-1)^{n+1} = 1-1+1-1+1-1+1-1+\cdots.$

```
(%i1)  a(n):=(-1)^(n+1)$
       rec:makelist(rectangle([i-1,0],[i,a(i)]),i,1,11)$
       draw2d(grid=true,xaxis=true,yaxis=true,
       xrange=[-.5,10.5],yrange=[-1.2,1.2],
       border=true,color=black,fill_color=red,rec)$
```



Number series

$\lim\limits_{n\to\infty} s_n = s \in R^*$ (if it exists) is called the **sum** of the series $\sum\limits_{n=1}^{\infty} a_n$, label $\sum\limits_{n=1}^{\infty} a_n = s$.

$\exists \lim\limits_{n\to\infty} s_n = s \in R$

Exists a finite limit,

$\sum\limits_{n=1}^{\infty} a_n$ converges to the sum s,

$\sum\limits_{n=1}^{\infty} a_n \longrightarrow s$, resp. $\sum\limits_{n=1}^{\infty} a_n = s$.

$\sum\limits_{n=1}^{\infty} a_n$ converges,

$\sum\limits_{n=1}^{\infty} a_n \longrightarrow$.

$\exists \lim\limits_{n\to\infty} s_n = \pm\infty$

Exists an infinite limit,

$\sum\limits_{n=1}^{\infty} a_n$ diverges to $\pm\infty$,

$\sum\limits_{n=1}^{\infty} a_n \longrightarrow \pm\infty$, resp. $\sum\limits_{n=1}^{\infty} a_n = \pm\infty$.

$\nexists \lim\limits_{n\to\infty} a_n$

Do not exists limit,

$\sum\limits_{n=1}^{\infty} a_n$ oscillates.

$\sum\limits_{n=1}^{\infty} a_n$ diverges,

$\sum\limits_{n=1}^{\infty} a_n \not\longrightarrow$.

Number series

$\lim\limits_{n\to\infty} s_n = s \in R^*$ (if it exists) is called the **sum** of the series $\sum\limits_{n=1}^{\infty} a_n$, label $\sum\limits_{n=1}^{\infty} a_n = s$.

$\exists \lim\limits_{n\to\infty} s_n = s \in R$ Exists a finite limit,

$\sum\limits_{n=1}^{\infty} a_n$ converges to the sum s,

$$\sum_{n=1}^{\infty} a_n \longrightarrow s, \text{ resp. } \sum_{n=1}^{\infty} a_n = s.$$

$\Bigg\}$ $\sum\limits_{n=1}^{\infty} a_n$ converges,

$$\sum_{n=1}^{\infty} a_n \longrightarrow.$$

$\exists \lim\limits_{n\to\infty} s_n = \pm\infty$ Exists an infinite limit,

$\sum\limits_{n=1}^{\infty} a_n$ diverges to $\pm\infty$,

$$\sum_{n=1}^{\infty} a_n \longrightarrow \pm\infty, \text{ resp. } \sum_{n=1}^{\infty} a_n = \pm\infty.$$

$\nexists \lim\limits_{n\to\infty} a_n$ Do not exists limit,

$\sum\limits_{n=1}^{\infty} a_n$ oscillates.

$\Bigg\}$ $\sum\limits_{n=1}^{\infty} a_n$ diverges,

$$\sum_{n=1}^{\infty} a_n \not\longrightarrow.$$

Number series

A series $\sum\limits_{n=1}^{\infty} a_n$.

- Changing the finite number (replacement, omission, addition, etc.) of members of the series $\sum\limits_{n=1}^{\infty} a_n$ does not affect the convergence, or series divergence.
- But it has an effect on his sum.

Harmonic series.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = \infty.$$

- $\sum\limits_{n=1}^{\infty} \frac{1}{n} = \infty.$ (The Harmonic series has an infinite sum.)
- $\sum\limits_{n=1}^{\infty} \frac{1}{n} \longrightarrow \infty.$ (The Harmonic series diverges to infinity.)

Number series

A series $\sum\limits_{n=1}^{\infty} a_n$.

- Changing the finite number (replacement, omission, addition, etc.) of members of the series $\sum\limits_{n=1}^{\infty} a_n$ does not affect the convergence, or series divergence.
- But it has an effect on his sum.

Harmonic series.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = \infty.$$

- $\sum\limits_{n=1}^{\infty} \frac{1}{n} = \infty.$ (The Harmonic series has an infinite sum.)
- $\sum\limits_{n=1}^{\infty} \frac{1}{n} \longrightarrow \infty.$ (The Harmonic series diverges to infinity.)

Number series

Some rules do not apply for infinite series, e.g. associative law:

$$\sum_{n=1}^{\infty}(-1)^{n+1} = \begin{cases} (1-1)+(1-1)+(1-1)+\cdots = 0+0+0+\cdots = 0, \\ 1+(-1+1)+(-1+1)+\cdots \quad = 1+0+0+\cdots = 1. \end{cases}$$

Geometric series.

$$\sum_{n=1}^{\infty} q^{n-1} = \sum_{n=0}^{\infty} q^{n} = 1+q+q^2+\cdots = \frac{1}{1-q} \text{ for all } q \in (-1;1).$$

- $\sum\limits_{n=1}^{\infty} q^{n-1} = \lim\limits_{n\to\infty} s_n$, $s_n = (1+q+\cdots+q^{n-1})\frac{1-q}{1-q} = \frac{1-q^n}{1-q} = \frac{\frac{1}{q}-q^{n-1}}{\frac{1}{q}-1}$ for $q \neq 1$.

$$S = \sum_{n=1}^{\infty} q^{n-1} = \lim_{n\to\infty} s_n = \begin{cases} \frac{q^n-1}{q-1} \to \frac{\infty-1}{q-1} = \infty. & \Rightarrow \bullet\ S = \infty & \text{for } q > 1. \\ 1+1+1+1+\cdots \to \infty. & \Rightarrow \bullet\ S = \infty & \text{for } q = 1. \\ \frac{q^n-1}{q-1} \to \frac{0-1}{q-1} = \frac{1}{1-q}. & \Rightarrow \bullet\ S = \frac{1}{1-q} & \text{for } q \in (-1;1). \\ -1+1-1+1-1+\cdots. & \Rightarrow \bullet\ \nexists & \text{for } q = -1. \\ \frac{\frac{1}{q}-q^{n-1}}{\frac{1}{q}-1},\ \frac{1}{q} \to 0,\ q^{n-1} \to \pm\infty. & \Rightarrow \bullet\ \nexists & \text{for } q < -1. \end{cases}$$

Number series

Some rules do not apply for infinite series, e.g. associative law:

$$\sum_{n=1}^{\infty}(-1)^{n+1}=\begin{cases}(1-1)+(1-1)+(1-1)+\cdots=0+0+0+\cdots=0,\\ 1+(-1+1)+(-1+1)+\cdots \quad =1+0+0+\cdots=1.\end{cases}$$

Geometric series.

$$\sum_{n=1}^{\infty}q^{n-1}=\sum_{n=0}^{\infty}q^{n}=1+q+q^2+\cdots=\tfrac{1}{1-q} \text{ for all } q\in(-1;1).$$

- $\sum_{n=1}^{\infty}q^{n-1}=\lim_{n\to\infty}s_n$, $s_n=(1+q+\cdots+q^{n-1})\frac{1-q}{1-q}=\frac{1-q^n}{1-q}=\frac{\frac{1}{q}-q^{n-1}}{\frac{1}{q}-1}$ for $q\neq 1$.

$$S=\sum_{n=1}^{\infty}q^{n-1}=\lim_{n\to\infty}s_n=\begin{cases}\frac{q^n-1}{q-1}\to\frac{\infty-1}{q-1}=\infty. & \Rightarrow \bullet\ S=\infty \quad \text{for } q>1.\\ 1+1+1+1+\cdots\to\infty. & \Rightarrow \bullet\ S=\infty \quad \text{for } q=1.\\ \frac{q^n-1}{q-1}\to\frac{0-1}{q-1}=\frac{1}{1-q}. & \Rightarrow \bullet\ S=\frac{1}{1-q} \text{ for } q\in(-1;1).\\ -1+1-1+1-1+\cdots. & \Rightarrow \bullet\ \nexists \quad \text{for } q=-1.\\ \frac{\frac{1}{q}-q^{n-1}}{\frac{1}{q}-1},\ \frac{1}{q}\to 0,\ q^{n-1}\to\pm\infty. & \Rightarrow \bullet\ \nexists \quad \text{for } q<-1.\end{cases}$$

Number series

```
(%i4) sq(q):=sum(q^n,n,1,inf)$ sq(1/2),simpsum;
      sq(1/3),simpsum; sq(-1/2),simpsum; sq(2),simpsum;
```

(%o1) 1

(%o2) $\frac{1}{2}$

(%o3) $-\frac{1}{3}$

```
(%o4) sum: sum is divergent.
      #0: sq(q=2) -- an error. To debug this try: debugmode(true);
```

- It is enough to change the value of q at the beginning in the following example.

```
(%i1) q:0.8$ a(n,q):=q^n$ peca:makelist([i,a(i,q)],i,1,11)$
      reca:makelist(rectangle([i-1,0],[i,a(i,q)]),i,1,11)$
      draw2d(grid=true,xaxis=true,yaxis=true,
      xrange=[-.5,10.5],yrange=[-4,4],
      border=true,color=black,fill_color=light_red,reca,
      label([concat("q=",string(q)),3,3.5]),
      color=blue,explicit(a(n,q),n,1,11),
      point_type=7,color=blue,points(peca))$
```

Number series

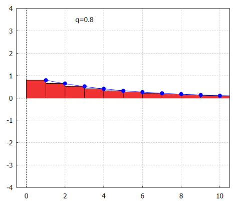
```
(%i4)  sq(q):=sum(q^n,n,1,inf)$ sq(1/2),simpsum;
       sq(1/3),simpsum; sq(-1/2),simpsum; sq(2),simpsum;
(%o1)  1
(%o2)  1/2
(%o3)  -1/3
(%o4)  sum:  sum is divergent.
       #0:  sq(q=2) -- an error.  To debug this try:  debugmode(true);
```

- It is enough to change the value of q at the beginning in the following example.

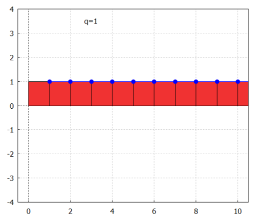
```
(%i1)  q:0.8$ a(n,q):=q^n$ peca:makelist([i,a(i,q)],i,1,11)$
       reca:makelist(rectangle([i-1,0],[i,a(i,q)]),i,1,11)$
       draw2d(grid=true,xaxis=true,yaxis=true,
       xrange=[-.5,10.5],yrange=[-4,4],
       border=true,color=black,fill_color=light_red,reca,
       label([concat("q=",string(q)),3,3.5]),
       color=blue,explicit(a(n,q),n,1,11),
       point_type=7,color=blue,points(peca))$
```

Number series

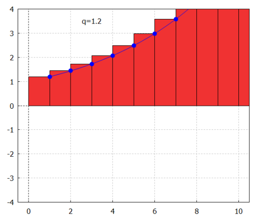
The commands will display the following graphs:



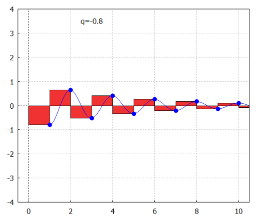
$q = 0.8$



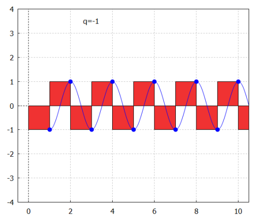
$q = 1$



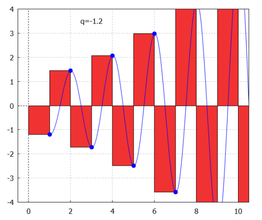
$q = 1.2$



$q = -0.8$



$q = -1$



$q = -1.2$

Number series

Necessary convergence condition.

- $\sum\limits_{n=1}^{\infty} a_n \longrightarrow.$ $\Rightarrow$ • $\lim\limits_{n\to\infty} a_n = 0.$

- $\lim\limits_{n\to\infty} a_n \neq 0$ (the limit does not exist or there is a non-zero).

  $\Rightarrow$ • $\sum\limits_{n=1}^{\infty} a_n \not\longrightarrow$ (oscillates or $\longrightarrow \pm\infty$).

The Harmonic series $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges to infinity.

- $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = \infty.$
- $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{n} = 0.$

The Geometric series $\sum\limits_{n=1}^{\infty} \frac{1}{2^n}$ with $q = \frac{1}{2}$ konverges to 2.

- $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2.$
- $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{2^n} = 0.$

Number series

Necessary convergence condition.

- $\sum\limits_{n=1}^{\infty} a_n \longrightarrow.$ $\Rightarrow$ • $\lim\limits_{n\to\infty} a_n = 0.$

- $\lim\limits_{n\to\infty} a_n \neq 0$ (the limit does not exist or there is a non-zero).

  $\Rightarrow$ • $\sum\limits_{n=1}^{\infty} a_n \not\longrightarrow$ (oscillates or $\longrightarrow \pm\infty$).

The Harmonic series $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges to infinity.

- $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = \infty.$
- $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{n} = 0.$

The Geometric series $\sum\limits_{n=1}^{\infty} \frac{1}{2^n}$ with $q = \frac{1}{2}$ konverges to 2.

- $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2.$
- $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{2^n} = 0.$

Number series

The series $\sum\limits_{n=1}^{\infty} a_n$, $a_n \geq 0$, $n \in N$ (**non-negative terms**) always has a sum $0 \leq s = \sum\limits_{n=1}^{\infty} a_n \leq \infty$.

Direct comparison test. $0 \leq a_n \leq b_n$, $n \in N$.

- $\sum\limits_{n=1}^{\infty} b_n \longrightarrow .$ $\Rightarrow$ • $\sum\limits_{n=1}^{\infty} a_n \longrightarrow .$
- $\sum\limits_{n=1}^{\infty} a_n \longrightarrow \infty.$ $\Rightarrow$ • $\sum\limits_{n=1}^{\infty} b_n \longrightarrow \infty.$

Comparison test (limit form). $0 \leq a_n \leq b_n$, $n \in N$.

$\lim\limits_{n\to\infty} \frac{a_n}{b_n} = p$, $0 < p < \infty$.

- $\sum\limits_{n=1}^{\infty} a_n \longrightarrow .$ $\Leftrightarrow$ • $\sum\limits_{n=1}^{\infty} b_n \longrightarrow .$
- $\sum\limits_{n=1}^{\infty} a_n \longrightarrow \infty.$ $\Leftrightarrow$ • $\sum\limits_{n=1}^{\infty} b_n \longrightarrow \infty.$

Number series

Ratio test (d'Alembert's ratio test). $a_n > 0,\ n\in N.$

- $\frac{a_{n+1}}{a_n} \le q < 1,\ n\in N$, where $q\in(0;1)$. $\Rightarrow$ • $\sum\limits_{n=1}^{\infty} a_n \longrightarrow.$
- $1 \le \frac{a_{n+1}}{a_n},\ n\in N.$ $\Rightarrow$ • $\sum\limits_{n=1}^{\infty} a_n \longrightarrow \infty.$

d'Alembert's ratio test (limit form). $a_n > 0,\ n\in N.$

$\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = p.$

- $p < 1.$ $\Rightarrow$ • $\sum\limits_{n=1}^{\infty} a_n \longrightarrow.$
- $p > 1.$ $\Rightarrow$ • $\sum\limits_{n=1}^{\infty} a_n \not\longrightarrow.$

For $p = 1$ we cannot decide.

- $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} n \longrightarrow \infty$, but $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{n+1}{n} = 1.$
- $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \frac{1}{n^2} \longrightarrow$, but $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim\limits_{n\to\infty} \frac{n^2}{(n+1)^2} = \lim\limits_{n\to\infty} \frac{n^2}{n^2+2n+1} = 1.$

Number series

Ratio test (d'Alembert's ratio test). $a_n > 0,\ n \in N.$

- $\frac{a_{n+1}}{a_n} \leq q < 1,\ n \in N$, where $q \in (0;1)$. $\Rightarrow$ • $\sum\limits_{n=1}^{\infty} a_n \longrightarrow.$
- $1 \leq \frac{a_{n+1}}{a_n},\ n \in N.$ $\Rightarrow$ • $\sum\limits_{n=1}^{\infty} a_n \longrightarrow \infty.$

d'Alembert's ratio test (limit form). $a_n > 0,\ n \in N.$

$\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = p.$

- $p < 1.\ \Rightarrow$ • $\sum\limits_{n=1}^{\infty} a_n \longrightarrow.$
- $p > 1.\ \Rightarrow$ • $\sum\limits_{n=1}^{\infty} a_n \not\longrightarrow.$

For $p = 1$ we cannot decide.

- $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} n \longrightarrow \infty$, but $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{n+1}{n} = 1.$
- $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \frac{1}{n^2} \longrightarrow$, but $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim\limits_{n\to\infty} \frac{n^2}{(n+1)^2} = \lim\limits_{n\to\infty} \frac{n^2}{n^2+2n+1} = 1.$

Number series

Root test (Cauchy root test). $a_n \geq 0,\ n \in N$.

- $\sqrt[n]{a_n} \leq q < 1,\ n \in N$, where $q \in (0;1)$. $\Rightarrow$ • $\sum\limits_{n=1}^{\infty} a_n \longrightarrow$.
- $1 \leq \sqrt[n]{a_n},\ n \in N$. $\Rightarrow$ • $\sum\limits_{n=1}^{\infty} a_n \longrightarrow \infty$.

Cauchy root test (limit form). $a_n \geq 0,\ n \in N$.

$\lim\limits_{n\to\infty} \sqrt[n]{a_n} = p$.

- $p < 1$. $\Rightarrow$ • $\sum\limits_{n=1}^{\infty} a_n \longrightarrow$.
- $p > 1$. $\Rightarrow$ • $\sum\limits_{n=1}^{\infty} a_n \not\longrightarrow$.

For $p = 1$ we cannot decide.

- $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} n \longrightarrow \infty$, but $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = \lim\limits_{n\to\infty} \sqrt[n]{n} = 1$.
- $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \frac{1}{n^2} \longrightarrow$, but $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = \lim\limits_{n\to\infty} \sqrt[n]{\frac{1}{n^2}} = \lim\limits_{n\to\infty} \frac{1}{\sqrt[n]{n^2}} = \lim\limits_{n\to\infty} \frac{1}{\sqrt[n]{n}\sqrt[n]{n}} = 1$.

Number series

Root test (Cauchy root test). $a_n \geq 0,\ n \in N.$

- $\sqrt[n]{a_n} \leq q < 1,\ n \in N$, where $q \in (0;1)$. $\Rightarrow$ • $\sum\limits_{n=1}^{\infty} a_n \longrightarrow.$
- $1 \leq \sqrt[n]{a_n},\ n \in N.$ $\Rightarrow$ • $\sum\limits_{n=1}^{\infty} a_n \longrightarrow \infty.$

Cauchy root test (limit form). $a_n \geq 0,\ n \in N.$

$\lim\limits_{n\to\infty} \sqrt[n]{a_n} = p.$

- $p < 1.$ $\Rightarrow$ • $\sum\limits_{n=1}^{\infty} a_n \longrightarrow.$
- $p > 1.$ $\Rightarrow$ • $\sum\limits_{n=1}^{\infty} a_n \not\longrightarrow.$

For $p = 1$ we cannot decide.

- $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} n \longrightarrow \infty$, but $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = \lim\limits_{n\to\infty} \sqrt[n]{n} = 1.$
- $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \frac{1}{n^2} \longrightarrow$, but $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = \lim\limits_{n\to\infty} \sqrt[n]{\frac{1}{n^2}} = \lim\limits_{n\to\infty} \frac{1}{\sqrt[n]{n^2}} = \lim\limits_{n\to\infty} \frac{1}{\sqrt[n]{n}\sqrt[n]{n}} = 1.$

Number series

$\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \frac{a^n}{n!} \longrightarrow$ for $a > 0$.

d'Alembert's ratio test:

- $\lim\limits_{n\to\infty} \frac{a^{n+1}}{(n+1)!} \frac{n!}{a^n} = \lim\limits_{n\to\infty} \frac{a}{n+1} = \frac{a}{\infty} = 0 < 1.$ $\Rightarrow$ • $\sum\limits_{n=1}^{\infty} \frac{a^n}{n!} \longrightarrow$ for $a > 0$.

Cauchy root test:

- $\lim\limits_{n\to\infty} \sqrt[n]{\frac{a^n}{n!}} = \lim\limits_{n\to\infty} \frac{a}{\sqrt[n]{n!}} = \frac{a}{\infty} = 0 < 1.$ $\Rightarrow$ • $\sum\limits_{n=1}^{\infty} \frac{a^n}{n!} \longrightarrow$ for $a > 0$.

```
(%i5)  an(n,a):=a^n/n!$ a:2$ limit(an(n,a),n,inf,plus);
            limit(an(n+1,a)/an(n,a),n,inf,plus);
            limit((an(n,a))^(1/n),n,inf,plus);
(%o3)  0
(%o4)  0
(%o5)  0
```

Number series

- If $\sum\limits_{n=1}^{\infty} a_n \longrightarrow$ and $\sum\limits_{n=1}^{\infty} |a_n| \longrightarrow$, then $\sum\limits_{n=1}^{\infty} a_n$ **converges absolutely**, label $\sum\limits_{n=1}^{\infty} a_n \xrightarrow{\text{A}}$.
- If $\sum\limits_{n=1}^{\infty} a_n \longrightarrow$ and $\sum\limits_{n=1}^{\infty} |a_n| \not\longrightarrow$, then $\sum\limits_{n=1}^{\infty} a_n$ **converges relatively**, label $\sum\limits_{n=1}^{\infty} a_n \xrightarrow{\text{R}}$.

The series $\sum\limits_{n=1}^{\infty} |a_n|$, $a_n \in R$, $n \in N$ always has a sum $0 \leq s = \sum\limits_{n=1}^{\infty} |a_n| \leq \infty$.

- $\sum\limits_{n=1}^{\infty} |a_n| \xrightarrow{\text{A}}$, i.e. $\sum\limits_{n=1}^{\infty} |a_n| \longrightarrow$. $\Rightarrow$ • $\sum\limits_{n=1}^{\infty} a_n \longrightarrow$.

Alternating series test (Leibniz criterion).

$$\left.\begin{array}{l} \bullet\ a_n \geq 0,\ n \in N,\ \{a_n\}_{n=1}^{\infty} \text{ is non-increasing.} \\ \bullet\ \lim\limits_{n\to\infty} a_n = 0. \end{array}\right\} \Rightarrow \bullet\ \sum_{n=1}^{\infty} (-1)^{n+1} a_n \longrightarrow.$$

- The series $\sum\limits_{n=1}^{\infty} (-1)^{n+1} a_n$, where $a_n \geq 0$ or $a_n \leq 0$ is called an **alternating series**.

Number series

- If $\sum\limits_{n=1}^{\infty} a_n \longrightarrow$ and $\sum\limits_{n=1}^{\infty} |a_n| \longrightarrow$, then $\sum\limits_{n=1}^{\infty} a_n$ **converges absolutely**, label $\sum\limits_{n=1}^{\infty} a_n \xrightarrow{A}$.
- If $\sum\limits_{n=1}^{\infty} a_n \longrightarrow$ and $\sum\limits_{n=1}^{\infty} |a_n| \not\longrightarrow$, then $\sum\limits_{n=1}^{\infty} a_n$ **converges relatively**, label $\sum\limits_{n=1}^{\infty} a_n \xrightarrow{R}$.

The series $\sum\limits_{n=1}^{\infty} |a_n|$, $a_n \in R$, $n \in N$ always has a sum $0 \leq s = \sum\limits_{n=1}^{\infty} |a_n| \leq \infty$.

- $\sum\limits_{n=1}^{\infty} |a_n| \xrightarrow{A}$, i.e. $\sum\limits_{n=1}^{\infty} |a_n| \longrightarrow$. $\Rightarrow$ • $\sum\limits_{n=1}^{\infty} a_n \longrightarrow$.

Alternating series test (Leibniz criterion).

$$\left.\begin{array}{l} \bullet\ a_n \geq 0,\ n \in N,\ \{a_n\}_{n=1}^{\infty} \text{ is non-increasing.} \\ \bullet\ \lim\limits_{n\to\infty} a_n = 0. \end{array}\right\} \Rightarrow \bullet \sum_{n=1}^{\infty} (-1)^{n+1} a_n \longrightarrow.$$

- The series $\sum\limits_{n=1}^{\infty} (-1)^{n+1} a_n$, where $a_n \geq 0$ or $a_n \leq 0$ is called an **alternating series**.

Number series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n+1}}{n} + \cdots \xrightarrow{\text{R}}.$$ (Anharmonic series.)

Alternating series test (Leibniz criterion):

- $a_n = \frac{1}{n} > 0$, $n \in N$, $\{a_n\}_{n=1}^{\infty} = \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is decreasing (non-increasing).
- $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{n} = \frac{1}{\infty} = 0.$

$$\Rightarrow \bullet \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n+1}}{n} + \cdots \xrightarrow{\text{R}}.$$

Anharmonic series and Harmonic series.

- $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n+1}}{n} + \cdots = \ln 2$ (Anharmonic series).
- $\sum\limits_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots = \infty$ (Harmonic series).

Number series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n+1}}{n} + \cdots \xrightarrow{R}.$$ (Anharmonic series.)

Alternating series test (Leibniz criterion):

- $a_n = \frac{1}{n} > 0,\ n \in N,\ \{a_n\}_{n=1}^{\infty} = \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is decreasing (non-increasing).
- $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{n} = \frac{1}{\infty} = 0.$

$$\Rightarrow \bullet \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n+1}}{n} + \cdots \xrightarrow{R}.$$

Anharmonic series and Harmonic series.

- $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n+1}}{n} + \cdots = \ln 2$ (Anharmonic series).
- $\sum\limits_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots = \infty$ (Harmonic series).

Functions

The function $y = f(x)$, $x \in D(f)$, i.e. $f\colon\ D(f) \to H(f)$.

- $D(f) \subset R$ A function of a real variable.
- $H(f) \subset R$ A real function.

Explicit form: • $y = f(x)$, $x \in D(f)$ (Analytical formula).

Parametric form: • $f\colon\ x = \varphi(t)$, $y = \psi(t)$, $t \in J$, $J \subset R$ (Auxiliary functions φ, ψ).

Implicit form: • $f\colon\ F(x, y) = 0$, conditions for $[x; y]$ (Implicit equation).

The function $f\colon\ y = |x|$, $x \in R$.

We can define the function $f\colon\ y = |x|$, $x \in R$, for example:

Explicit form: • $y = \sqrt{x^2}$, resp. • $y = \max\{-x, x\}$.

Parametric form: • $x = t$, $y = |t|$, $t \in R$, resp. • $x = t$, $y = \sqrt{t^2}$, $t \in R$.

Implicit form: • $y^2 - x^2 = 0$, $y \geq 0$, resp. • $y - |x| = 0$.

Functions

The function $y = f(x)$, $x \in D(f)$, i.e. $f\colon\ D(f) \to H(f)$.

- $D(f) \subset R$ A function of a real variable.
- $H(f) \subset R$ A real function.

Explicit form:
- $y = f(x),\ x \in D(f)$ (Analytical formula).

Parametric form:
- $f\colon\ x = \varphi(t),\ y = \psi(t),\ t \in J,\ J \subset R$ (Auxiliary functions φ, ψ).

Implicit form:
- $f\colon\ F(x, y) = 0$, conditions for $[x; y]$ (Implicit equation).

The function $f\colon\ y = |x|,\ x \in R$.

We can define the function $f\colon\ y = |x|,\ x \in R$, for example:

Explicit form:
- $y = \sqrt{x^2}$, resp.
- $y = \max\{-x, x\}$.

Parametric form:
- $x = t,\ y = |t|,\ t \in R$, resp.
- $x = t,\ y = \sqrt{t^2},\ t \in R$.

Implicit form:
- $y^2 - x^2 = 0,\ y \geq 0$, resp.
- $y - |x| = 0$.

Functions

The function $y = f(x)$, $x \in D(f)$, i.e. $f\colon\ D(f) \to H(f)$.

- $D(f) \subset R$ A function of a real variable.
- $H(f) \subset R$ A real function.

Explicit form: • $y = f(x),\ x \in D(f)$ (Analytical formula).

Parametric form: • $f\colon\ x = \varphi(t),\ y = \psi(t),\ t \in J,\ J \subset R$ (Auxiliary functions φ, ψ).

Implicit form: • $f\colon\ F(x, y) = 0$, conditions for $[x; y]$ (Implicit equation).

The function $f\colon\ y = |x|,\ x \in R$.

We can define the function $f\colon\ y = |x|$, $x \in R$, for example:

Explicit form: • $y = \sqrt{x^2}$, resp. • $y = \max\{-x, x\}$.

Parametric form: • $x = t,\ y = |t|,\ t \in R$, resp. • $x = t,\ y = \sqrt{t^2},\ t \in R$.

Implicit form: • $y^2 - x^2 = 0,\ y \geq 0$, resp. • $y - |x| = 0$.

Functions

A function $y = f(x)$, $x \in D(f)$ and a set $A \subset D(f)$.

• $\forall x \in A\colon\ a \le f(x)$	a is the lower bound,	f is bounded from below	on the set A.
• $\forall x \in A\colon\ f(x) \le b$	b is an upper bound,	f is bounded from above	
• f is bounded from below and above		f is bounded	

• not bounded from below on the set A	f is unbounded from below	on the set A.
• not bounded from above on the set A	f is unbounded from above	
• not bounded on the set A	f is unbounded (below or above)	

A function $y = f(x)$, $x \in D(f)$ and a set $A \subset D(f)$.

- $A \neq D(f)$. A local property on the set A.
- $A = D(f)$. A global property (on the entire domain).

- $f\colon\ y = \sin x$ is bounded, i.e. is bounded on $D(f) = R$.
- $f\colon\ y = x^3$ is unbounded (from below and from above), f is bounded for example on $(0; 1\rangle$.

Functions

A function $y = f(x)$, $x \in D(f)$ and a set $A \subset D(f)$.

- $\forall x \in A$: $a \leq f(x)$ a is the lower bound, f is bounded from below
- $\forall x \in A$: $f(x) \leq b$ b is an upper bound, f is bounded from above
- f is bounded from below and above f is bounded

on the set A.

- not bounded from below on the set A f is unbounded from below
- not bounded from above on the set A f is unbounded from above
- not bounded on the set A f is unbounded (below or above)

on the set A.

A function $y = f(x)$, $x \in D(f)$ and a set $A \subset D(f)$.

- $A \neq D(f)$. A local property on the set A.
- $A = D(f)$. A global property (on the entire domain).

- f: $y = \sin x$ is bounded, i.e. is bounded on $D(f) = R$.
- f: $y = x^3$ is unbounded (from below and from above), f is bounded for example on $(0; 1\rangle$.

Functions

A function $y = f(x)$, $x \in D(f)$ and a set $A \subset D(f)$.

$\forall x \in A$: $a \leq f(x)$	a is the lower bound,	f is bounded from below	on the set A.
$\forall x \in A$: $f(x) \leq b$	b is an upper bound,	f is bounded from above	
f is bounded from below and above		f is bounded	

not bounded from below on the set A	f is unbounded from below	on the set A.
not bounded from above on the set A	f is unbounded from above	
not bounded on the set A	f is unbounded (below or above)	

A function $y = f(x)$, $x \in D(f)$ and a set $A \subset D(f)$.

- $A \neq D(f)$. A local property on the set A.
- $A = D(f)$. A global property (on the entire domain).

- f: $y = \sin x$ is bounded, i.e. is bounded on $D(f) = R$.
- f: $y = x^3$ is unbounded (from below and from above), f is bounded for example on $(0; 1\rangle$.

Functions

A function $y = f(x)$, $x \in D(f)$ and a set $A \subset D(f)$.

$$\left.\begin{array}{ll} \bullet\ \inf f(A) = \inf\{f(x), x \in A\} & \text{Local infimum} \\ \bullet\ \sup f(A) = \sup\{f(x), x \in A\} & \text{Local supremum} \end{array}\right\} \text{ on the set } A.$$

$$\left.\begin{array}{ll} \bullet\ \inf f(x) = \inf\{f(x), x \in D(f)\} & \text{Global infimum} \\ \bullet\ \sup f(x) = \sup\{f(x), x \in D(f)\} & \text{Global supremum} \end{array}\right\} \text{ (on the definition domain).}$$

Function $y = f(x)$, $x \in D(f)$, a set $A \subset D(f)$ and a point $x_0 \in A$.

$$\left.\begin{array}{lll} \bullet\ \forall x \in A: & f(x_0) \leq f(x) & \text{Minimum.} \\ & f(x_0) \geq f(x) & \text{Maximum.} \\ \bullet\ \forall x \in A,\ x \neq x_0: & f(x_0) < f(x) & \left.\begin{array}{l}\text{Strict minimum.} \\ \text{Strict maximum.}\end{array}\right\} \text{Strict extrema.} \\ & f(x_0) > f(x) & \end{array}\right\} \begin{array}{l}\text{Extrema} \\ \quad \text{on the set } A.\end{array}$$

$A \subset D(f)$, $A \neq D(f)$ • Local extrema on the set A.

$A = D(f)$ • Global (absolute) extrema (on the domain).

Functions

A function $y = f(x)$, $x \in D(f)$ and a set $A \subset D(f)$.

$$\left.\begin{array}{ll} \bullet\ \inf f(A) = \inf\{f(x), x \in A\} & \text{Local infimum} \\ \bullet\ \sup f(A) = \sup\{f(x), x \in A\} & \text{Local supremum} \end{array}\right\} \text{ on the set } A.$$

$$\left.\begin{array}{ll} \bullet\ \inf f(x) = \inf\{f(x), x \in D(f)\} & \text{Global infimum} \\ \bullet\ \sup f(x) = \sup\{f(x), x \in D(f)\} & \text{Global supremum} \end{array}\right\} \text{ (on the definition domain).}$$

Function $y = f(x)$, $x \in D(f)$, a set $A \subset D(f)$ and a point $x_0 \in A$.

$$\left.\begin{array}{lll} \bullet\ \forall x \in A: & f(x_0) \le f(x) & \text{Minimum.} \\ & f(x_0) \ge f(x) & \text{Maximum.} \\ \bullet\ \forall x \in A,\ x \ne x_0: & \begin{array}{l} f(x_0) < f(x) \\ f(x_0) > f(x) \end{array} & \left.\begin{array}{l} \text{Strict minimum.} \\ \text{Strict maximum.} \end{array}\right\} \text{Strict extrema.} \end{array}\right\} \begin{array}{l} \text{Extrema} \\ \quad \text{on the set } A. \end{array}$$

$A \subset D(f)$, $A \ne D(f)$ • Local extrema on the set A.

$A = D(f)$ • Global (absolute) extrema (on the domain).

Functions

A function $y = f(x)$, $x \in D(f)$ and a set $A \subset D(f)$.

$$\left.\begin{array}{ll} \bullet\ \inf f(A) = \inf\{f(x), x \in A\} & \text{Local infimum} \\ \bullet\ \sup f(A) = \sup\{f(x), x \in A\} & \text{Local supremum} \end{array}\right\} \text{ on the set } A.$$

$$\left.\begin{array}{ll} \bullet\ \inf f(x) = \inf\{f(x), x \in D(f)\} & \text{Global infimum} \\ \bullet\ \sup f(x) = \sup\{f(x), x \in D(f)\} & \text{Global supremum} \end{array}\right\} \text{ (on the definition domain).}$$

Function $y = f(x)$, $x \in D(f)$, a set $A \subset D(f)$ and a point $x_0 \in A$.

$$\left.\begin{array}{l} \begin{array}{lll} \bullet\ \forall x \in A: & f(x_0) \leq f(x) & \text{Minimum.} \\ & f(x_0) \geq f(x) & \text{Maximum.} \end{array} \\ \left.\begin{array}{lll} \bullet\ \forall x \in A,\ x \neq x_0: & f(x_0) < f(x) & \text{Strict minimum.} \\ & f(x_0) > f(x) & \text{Strict maximum.} \end{array}\right\} \text{Strict extrema.} \end{array}\right\} \begin{array}{l} \text{Extrema} \\ \quad \text{on the set } A. \end{array}$$

$A \subset D(f)$, $A \neq D(f)$ • Local extrema on the set A.

$A = D(f)$ • Global (absolute) extrema (on the domain).

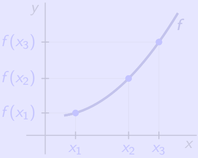
Functions

A function $y = f(x)$, $x \in D(f)$ and a set $A \subset D(f)$.

- $\forall x_1, x_2 \in A,\ x_1 < x_2$: $f(x_1) < f(x_2)$ Increasing.
  $f(x_1) > f(x_2)$ Descending.
- $\forall x_1, x_2 \in A,\ x_1 < x_2$: $f(x_1) \leq f(x_2)$ Non-decreasing.
  $f(x_1) \geq f(x_2)$ Non-increasing.
- $\forall x_1, x_2 \in A$: $f(x_1) = f(x_2)$ Constant.

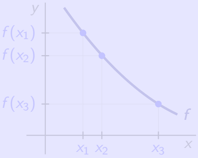
Increasing, Descending: Strictly monotonic.

All: Monotonic on the set A.



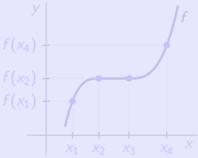
$f(x_1) < f(x_2) < f(x_3)$

Increasing function



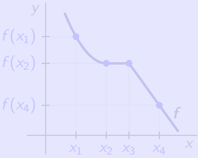
$f(x_1) > f(x_2) > f(x_3)$

Decreasing function



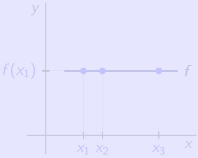
$f(x_1) < f(x_2) = f(x_3) < f(x_4)$

Non-decreasing function



$f(x_1) > f(x_2) = f(x_3) > f(x_4)$

Non-increasing function



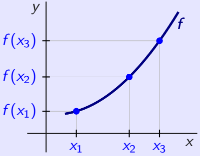
$f(x_1) = f(x_2) = f(x_3)$

Constant function

Functions

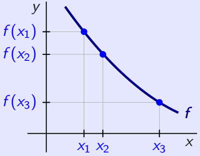
A function $y = f(x)$, $x \in D(f)$ and a set $A \subset D(f)$.

$$\left.\begin{array}{l}
\left.\begin{array}{lll}
\bullet\ \forall x_1, x_2 \in A,\ x_1 < x_2: & f(x_1) < f(x_2) & \text{Increasing.} \\
 & f(x_1) > f(x_2) & \text{Descending.}
\end{array}\right\}\text{Strictly monotonic.} \\
\begin{array}{lll}
\bullet\ \forall x_1, x_2 \in A,\ x_1 < x_2: & f(x_1) \leq f(x_2) & \text{Non-decreasing.} \\
 & f(x_1) \geq f(x_2) & \text{Non-increasing.} \\
\bullet\ \forall x_1, x_2 \in A: & f(x_1) = f(x_2) & \text{Constant.}
\end{array}
\end{array}\right\}\begin{array}{l}\text{Monotonic} \\ \quad\text{on the set } A.\end{array}$$



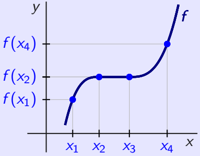
$f(x_1) < f(x_2) < f(x_3)$

Increasing function



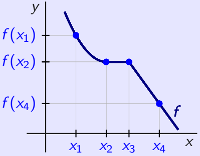
$f(x_1) > f(x_2) > f(x_3)$

Decreasing function



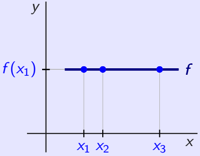
$f(x_1) < f(x_2) = f(x_3) < f(x_4)$

Non-decreasing function



$f(x_1) > f(x_2) = f(x_3) > f(x_4)$

Non-increasing function



$f(x_1) = f(x_2) = f(x_3)$

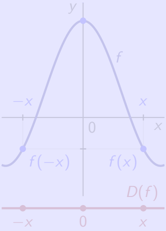
Constant function

Functions

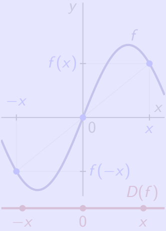
A function $y = f(x)$, $x \in D(f)$.

- $\forall x \in D(f)$: $-x \in D(f)$, $f(x) = f(-x)$ Even function.

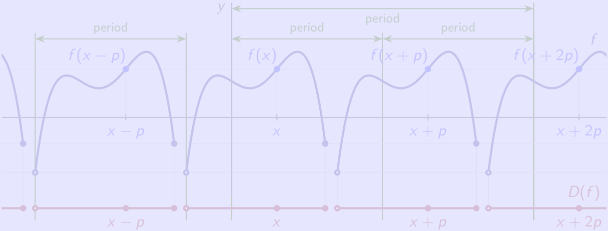
  $f(x) = -f(-x)$ Odd function.
- $\forall x \in D(f)$: $x \pm p \in D(f)$, $f(x) = f(x \pm p)$, $p \in R - \{0\}$ Periodic function, p is the period.



Even function



Odd function



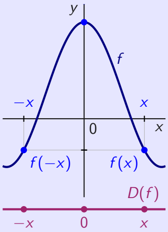
Periodic function

Functions

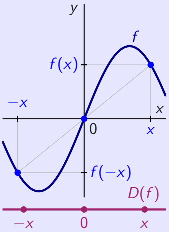
A function $y = f(x)$, $x \in D(f)$.

- $\forall x \in D(f)\colon\ -x \in D(f),\quad f(x) = \ \ f(-x)$ Even function.

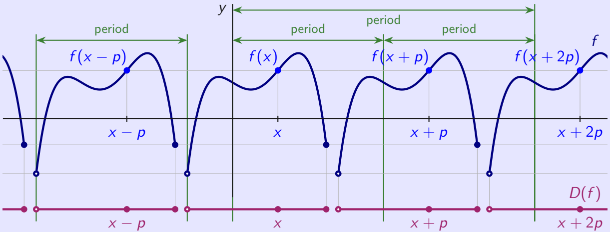
  $f(x) = -f(-x)$ Odd function.
- $\forall x \in D(f)\colon\ x \pm p \in D(f),\ f(x) = f(x \pm p),\ p \in R - \{0\}$ Periodic function, p is the period.



Even function



Odd function



Periodic function

Functions

A function $y = f(x)$, $x \in D(f)$, $I \subset D(f)$ is an interval, points $x_1, x_2 \in I$, $x_1 < x_2$.

- The line $p(x) = \frac{x_2 - x}{x_2 - x_1} f(x_1) + \frac{x - x_1}{x_2 - x_1} f(x_2)$, $x \in R$ connects the points $[x_1; f(x_1)]$ and $[x_2; f(x_2)]$.

$$\left.\begin{array}{lll} \bullet\ \forall x \in I,\ x_1 < x < x_2\colon & f(x) \leq p(x) & \text{Convex} \\ & f(x) < p(x) & \text{Strictly convex} \\ \bullet\ \forall x \in I,\ x_1 < x < x_2\colon & f(x) \geq p(x) & \text{Concave} \\ & f(x) > p(x) & \text{Strictly concave} \end{array}\right\} \text{on the interval } I.$$

A function $y = f(x)$, $x \in D(f)$,

$x_0 \in D(f)$ is the **inflection point** f (f has an **inflection** at the point x_0), if exists a neighborhood $O_\delta(x_0)$ such that the fuction f:

$$\left.\begin{array}{l} \bullet\ f \text{ is on } O_\delta^-(x_0) = (x_0 - \delta; x_0) \text{ strictly convex} \\ \bullet\ f \text{ is on } O_\delta^+(x_0) = (x_0; x_0 + \delta) \text{ strictly concave} \end{array}\right\} \text{resp.} \left\{\begin{array}{l} f \text{ is on } O_\delta^-(x_0) \text{ strictly concave.} \\ f \text{ is on } O_\delta^+(x_0) \text{ strictly convex.} \end{array}\right.$$

Functions

A function $y = f(x)$, $x \in D(f)$, $I \subset D(f)$ is an interval, points $x_1, x_2 \in I$, $x_1 < x_2$.

- The line $p(x) = \frac{x_2 - x}{x_2 - x_1} f(x_1) + \frac{x - x_1}{x_2 - x_1} f(x_2)$, $x \in R$ connects the points $[x_1; f(x_1)]$ and $[x_2; f(x_2)]$.

$$\left.\begin{array}{lll} \bullet\ \forall x \in I,\ x_1 < x < x_2: & f(x) \le p(x) & \text{Convex} \\ & f(x) < p(x) & \text{Strictly convex} \\ \bullet\ \forall x \in I,\ x_1 < x < x_2: & f(x) \ge p(x) & \text{Concave} \\ & f(x) > p(x) & \text{Strictly concave} \end{array}\right\} \text{ on the interval } I.$$

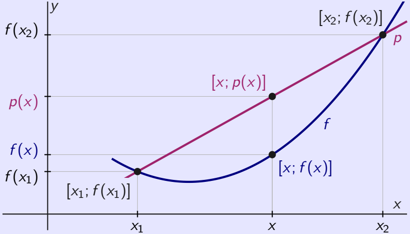
A function $y = f(x)$, $x \in D(f)$,

$x_0 \in D(f)$ is the **inflection point** f (f has an **inflection** at the point x_0), if exists a neighborhood $O_\delta(x_0)$ such that the fuction f:

$$\left.\begin{array}{l} \bullet\ f \text{ is on } O_\delta^-(x_0) = (x_0 - \delta; x_0) \text{ strictly convex} \\ \bullet\ f \text{ is on } O_\delta^+(x_0) = (x_0; x_0 + \delta) \text{ strictly concave} \end{array}\right\} \text{ resp. } \left\{\begin{array}{l} f \text{ is on } O_\delta^-(x_0) \text{ strictly concave.} \\ f \text{ is on } O_\delta^+(x_0) \text{ strictly convex.} \end{array}\right.$$

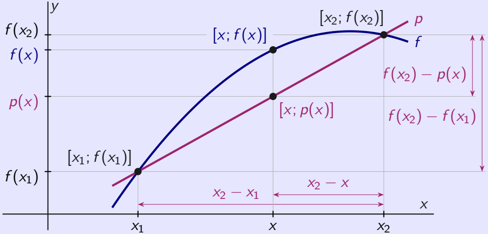
Functions

- A function $y = f(x)$, $x \in D(f)$, $I \subset D(f)$ is an interval, points $x_1, x_2 \in I$, $x_1 < x_2$.
- The line $p(x) = \frac{x_2 - x}{x_2 - x_1} f(x_1) + \frac{x - x_1}{x_2 - x_1} f(x_2)$, $x \in R$ connects the points $[x_1; f(x_1)]$, $[x_2; f(x_2)]$.



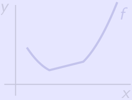
Convex function

$f(x) \leq p(x)$, $x_1 < x < x_2$

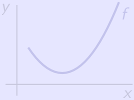


Concave function

$f(x) \geq p(x)$, $x_1 < x < x_2$



convex



Strictly convex

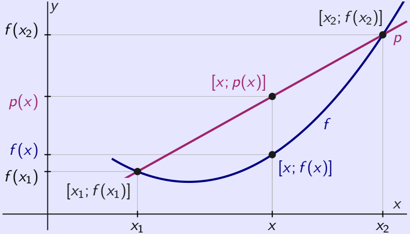
Convex and also Concave

Concave

Strictly concave

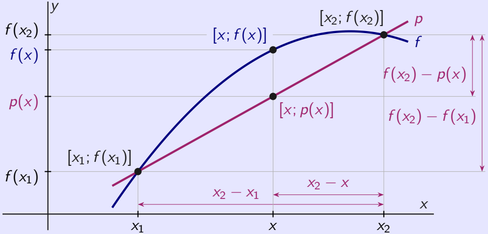
Functions

- A function $y = f(x)$, $x \in D(f)$, $I \subset D(f)$ is an interval, points $x_1, x_2 \in I$, $x_1 < x_2$.
- The line $p(x) = \frac{x_2 - x}{x_2 - x_1} f(x_1) + \frac{x - x_1}{x_2 - x_1} f(x_2)$, $x \in R$ connects the points $[x_1; f(x_1)]$, $[x_2; f(x_2)]$.



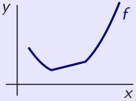
Convex function

$f(x) \leq p(x)$, $x_1 < x < x_2$

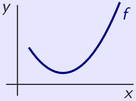


Concave function

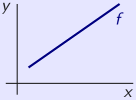
$f(x) \geq p(x)$, $x_1 < x < x_2$



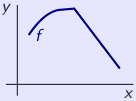
convex



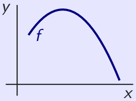
Strictly convex



Convex and also Concave



Concave



Strictly concave

Elementary Functions I

Elementary function is called each function created using the operations of addition, subtraction, multiplication, division or using composition of functions from basic elementary functions:

- $y = 1$,
- $y = x$,
- $y = e^x$,
- $y = \ln x$,
- $y = \sin x$,
- $y = \arcsin x$,
- $y = \arctan x$.

A polynomial of degree n is called

$$f_n\colon\ y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n, \text{ where } a_0, a_1, \ldots, a_n \in R,\ n \in N \cup \{0\},\ a_n \neq 0.$$

- $f_0\colon\ y = a_0,\ a_0 \neq 0$ is called a **constant function**.
- $f_1\colon\ y = a_0 + a_1x,\ a_1 \neq 0$ is called a **linear function**.
- $f_2\colon\ y = a_0 + a_1x + a_2x^2,\ a_2 \neq 0$ is called a **quadratic function**.

Elementary Functions I

Elementary function is called each function created using the operations of addition, subtraction, multiplication, division or using composition of functions from basic elementary functions:

- $y = 1$,
- $y = x$,
- $y = e^x$,
- $y = \ln x$,
- $y = \sin x$,
- $y = \arcsin x$,
- $y = \arctan x$.

A polynomial of degree n is called

$f_n\colon\ y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, where $a_0, a_1, \ldots, a_n \in R$, $n \in N \cup \{0\}$, $a_n \neq 0$.

- $f_0\colon\ y = a_0$, $a_0 \neq 0$ is called a **constant function**.
- $f_1\colon\ y = a_0 + a_1x$, $a_1 \neq 0$ is called a **linear function**.
- $f_2\colon\ y = a_0 + a_1x + a_2x^2$, $a_2 \neq 0$ is called a **quadratic function**.

Elementary Functions I

Rational fractional function is called

$f\colon\ y = \frac{f_n(x)}{f_m(x)} = \frac{a_0+a_1x+a_2x^2+\cdots+a_nx^n}{b_0+b_1x+b_2x^2+\cdots+b_mx^m}$, where f_n, f_m are polynomials of degrees $n, m \in N \cup \{0\}$.

Power function is called

$f\colon\ y = x^r$, where $r \in R$, $r \neq 0$.

Exponential function with base $a > 0$ is called

$f\colon\ y = a^x$, $x \in R$.

- The most important one is $f\colon\ y = \exp x = \mathrm{e}^x$ with base e (Euler's number).
- The graph is called the **exponential curve** and passes through the points $[0;1]$ and $[1;a]$.
- The graphs of the functions $y = a^x$, $y = a^{-x}$ are symmetric along the y axis.

Elementary Functions I

Rational fractional function is called

$f\colon\ y = \frac{f_n(x)}{f_m(x)} = \frac{a_0+a_1x+a_2x^2+\cdots+a_nx^n}{b_0+b_1x+b_2x^2+\cdots+b_mx^m}$, where f_n, f_m are polynomials of degrees $n, m \in N \cup \{0\}$.

Power function is called

$f\colon\ y = x^r$, where $r \in R,\ r \neq 0$.

Exponential function with base $a > 0$ is called

$f\colon\ y = a^x,\ x \in R.$

- The most important one is $f\colon\ y = \exp x = \mathrm{e}^x$ with base e (Euler's number).
- The graph is called the **exponential curve** and passes through the points $[0; 1]$ and $[1; a]$.
- The graphs of the functions $y = a^x$, $y = a^{-x}$ are symmetric along the y axis.

Elementary Functions I

Rational fractional function is called

$f\colon\ y = \frac{f_n(x)}{f_m(x)} = \frac{a_0+a_1x+a_2x^2+\cdots+a_nx^n}{b_0+b_1x+b_2x^2+\cdots+b_mx^m}$, where f_n, f_m are polynomials of degrees $n, m \in N \cup \{0\}$.

Power function is called

$f\colon\ y = x^r$, where $r \in R,\ r \neq 0$.

Exponential function with base $a > 0$ is called

$f\colon\ y = a^x,\ x \in R$.

- The most important one is $f\colon\ y = \exp x = \mathrm{e}^x$ with base e (Euler's number).
- The graph is called the **exponential curve** and passes through the points $[0;1]$ and $[1;a]$.
- The graphs of the functions $y = a^x$, $y = a^{-x}$ are symmetric along the y axis.

Elementary Functions I

Logarithmic function with base $a > 0,\ a \neq 1$ is called

$$f\colon\ y = \log_a x,\ x \in \langle 0; \infty).$$

- Logarithmic function $y = \log_a x,\ x \in \langle 0; \infty)$ is the inverse of the exponential function $y = a^x,\ x \in R$ with the same base $a > 0,\ a \neq 1$ $(y = \log_a x \Leftrightarrow x = a^y)$.
- For $a > 0,\ a \neq 1$ holds: $x = a^{\log_a x}$ for $x > 0$.
  $x = \log_a a^x$ for $x \in R$.
- The graph is called a logarithmic curve and passes through the points $[1; 0]$ and $[a; 1]$.
- The graphs of the functions $y = \log_a x$ and $y = \log_{a^{-1}} x$ are symmetric along the x axis.
- $a = 10.$ $\Rightarrow$ Decadal logarithm, label $\log x = \log_{10} x$.
- $a = \mathrm{e}.$ $\Rightarrow$ Natural logarithm, label $\ln x = \log_{\mathrm{e}} x$.
  `exp(x)=%e^x` and `log(x)` (natural logarithm) have the base e.
- If we want to calculate logarithm with another base, e.g. $\log_2 x$, we have to use construction $\log_2 x = \ln x / \ln 2$.

Elementary Functions I

Logarithmic function with base $a > 0,\ a \neq 1$ is called

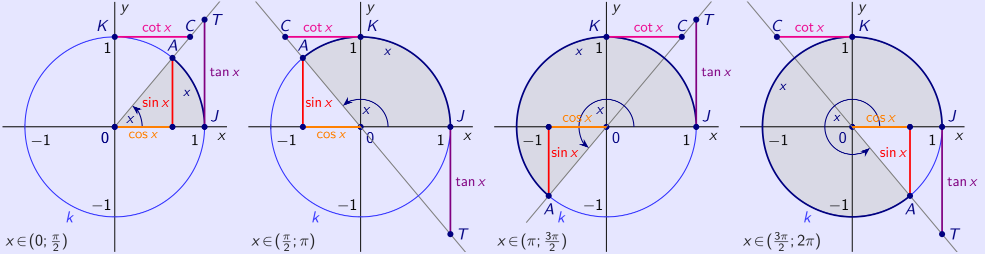
$$f\colon\ y = \log_a x,\ x \in \langle 0; \infty).$$

- Logarithmic function $y = \log_a x,\ x \in \langle 0; \infty)$ is the inverse of the exponential function $y = a^x,\ x \in R$ with the same base $a > 0,\ a \neq 1$ $(y = \log_a x \Leftrightarrow x = a^y)$.
- For $a > 0,\ a \neq 1$ holds: $x = a^{\log_a x}$ for $x > 0$.
  $x = \log_a a^x$ for $x \in R$.
- The graph is called a logarithmic curve and passes through the points $[1; 0]$ and $[a; 1]$.
- The graphs of the functions $y = \log_a x$ and $y = \log_{a^{-1}} x$ are symmetric along the x axis.
- $a = 10$. $\Rightarrow$ Decadal logarithm, label $\log x = \log_{10} x$.
- $a = \mathrm{e}$. $\Rightarrow$ Natural logarithm, label $\ln x = \log_{\mathrm{e}} x$.
  `exp(x)=%e^x` and `log(x)` (natural logarithm) have the base e.
- If we want to calculate logarithm with another base, e.g. $\log_2 x$, we have to use construction $\log_2 x = \ln x / \ln 2$.

Elementary Functions II

Trigonometric (goniometric) functions are:

- **Sine** $y = \sin x = |AA_x|$: $R \to \langle -1; 1 \rangle$.
- **Cosine** $y = \cos x = |OA_x|$: $R \to \langle -1; 1 \rangle$.
- **Tangent** $y = \tan x = \frac{\sin x}{\cos x} = |TJ|$: $R - \{\frac{\pi}{2} + k\pi, k \in Z\} \to R$.
- **Cotangent** $y = \cot x = \frac{\cos x}{\sin x} = |CK|$: $R - \{k\pi, k \in Z\} \to R$.

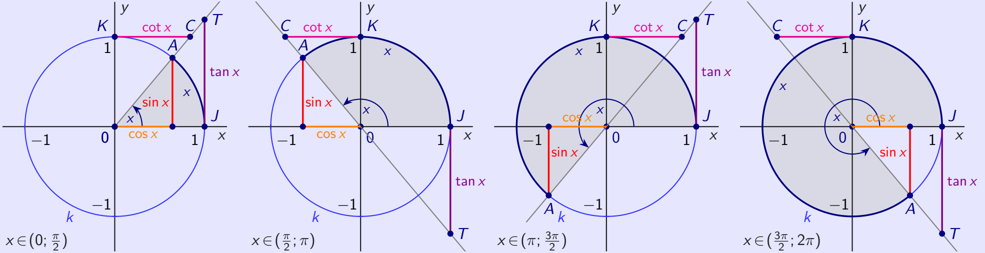


- The number π is called **Ludolf's**. Its value is approximately $3,141\,592\,654$.
- A circle with a radius $r = 1$ has a circumference of 2π.

Elementary Functions II

Trigonometric (goniometric) functions are:

- **Sine** $y = \sin x = |AA_x|$: $R \to \langle -1; 1 \rangle$.
- **Cosine** $y = \cos x = |OA_x|$: $R \to \langle -1; 1 \rangle$.
- **Tangent** $y = \tan x = \frac{\sin x}{\cos x} = |TJ|$: $R - \left\{ \frac{\pi}{2} + k\pi, k \in Z \right\} \to R$.
- **Cotangent** $y = \cot x = \frac{\cos x}{\sin x} = |CK|$: $R - \{ k\pi, k \in Z \} \to R$.



$x \in (0; \frac{\pi}{2})$ $x \in (\frac{\pi}{2}; \pi)$ $x \in (\pi; \frac{3\pi}{2})$ $x \in (\frac{3\pi}{2}; 2\pi)$

- The number π is called **Ludolf's**. Its value is approximately $3,141\,592\,654$.
- A circle with a radius $r = 1$ has a circumference of 2π.

Elementary Functions II

- In Maxima, trigonometric functions have the form `sin(x)`, `cos(x)`, `tan(x)`, `cot(x)`.
- Arguments of trigonometric functions must be entered in radians.
- If we want to use degrees, we must first convert to radians.

```
(%i3) tangrad(x):=tan(x/180*%pi); tangrad(22.5);
      ratsimp(tangrad(22.5));
```

(%o1) $\mathrm{tangrad}(x) := \tan\left(\frac{x}{180}\pi\right)$

(%o2) $\tan\left(0.125\pi\right)$

```
rat: replaced 0.125 by 1/8 = 0.125
```

(%o3) $\tan\left(\frac{\pi}{8}\right)$

- To simplify work with trigonometric functions, we can use commands `trigsimp`, `trigrat`, `trigexpand`, `trigreduce` and packages `atrig1`, `ntrig` or `spangl`, which contain additional support for working with trigonometric functions.
- We have to load the packages into the system using the command `load`.

Elementary Functions II

- In Maxima, trigonometric functions have the form `sin(x)`, `cos(x)`, `tan(x)`, `cot(x)`.
- Arguments of trigonometric functions must be entered in radians.
- If we want to use degrees, we must first convert to radians.

```
(%i3)  tangrad(x):=tan(x/180*%pi); tangrad(22.5);
       ratsimp(tangrad(22.5));
```

(%o1) $\mathrm{tangrad}(x) := \tan\left(\frac{x}{180}\pi\right)$

(%o2) $\tan\left(0.125\pi\right)$

```
rat:  replaced 0.125 by 1/8 = 0.125
```

(%o3) $\tan\left(\frac{\pi}{8}\right)$

- To simplify work with trigonometric functions, we can use commands `trigsimp`, `trigrat`, `trigexpand`, `trigreduce` and packages `atrig1`, `ntrig` or `spangl`, which contain additional support for working with trigonometric functions.
- We have to load the packages into the system using the command `load`.

Elementary Functions II

Sum formulas for sine and cosine. $x, y \in R$.

- $\sin(x \pm y) = \sin x \cdot \cos y \pm \cos x \cdot \sin y$.
- $\cos(x \pm y) = \cos x \cdot \cos y \mp \sin x \cdot \sin y$.
- $\sin 2x = \sin(x + x) = 2 \sin x \cdot \cos x$.
- $\cos 2x = \cos(x + x) = \cos^2 x - \sin^2 x$.
- $\sin^2 x = \frac{1 - \cos 2x}{2}$.
- $\cos^2 x = \frac{1 + \cos 2x}{2}$.
- $\sin^2 x + \cos^2 x = 1$.

Cyclometric functions are inverses of trigonometric functions:

Arcsine	$y = \arcsin x$:	$\langle -1; 1 \rangle \to \langle \frac{\pi}{2}; \frac{\pi}{2} \rangle$.
Arccosine	$y = \arccos x$:	$\langle -1; 1 \rangle \to \langle 0; \pi \rangle$.
Arctangent	$y = \arctan x$:	$R \to \left(-\frac{\pi}{2}; \frac{\pi}{2} \right)$.
Arccotangent	$y = \operatorname{arccot} x$:	$R \to (0; \pi)$.

- There are no inverse functions for trigonometric functions because they are not injective. It is necessary to narrow them appropriately.

Elementary Functions II

Sum formulas for sine and cosine. $x, y \in R$.

- $\sin(x \pm y) = \sin x \cdot \cos y \pm \cos x \cdot \sin y$.
- $\cos(x \pm y) = \cos x \cdot \cos y \mp \sin x \cdot \sin y$.
- $\sin 2x = \sin(x + x) = 2 \sin x \cdot \cos x$.
- $\cos 2x = \cos(x + x) = \cos^2 x - \sin^2 x$.
- $\sin^2 x = \frac{1 - \cos 2x}{2}$.
- $\cos^2 x = \frac{1 + \cos 2x}{2}$.
- $\sin^2 x + \cos^2 x = 1$.

Cyclometric functions are inverses of trigonometric functions:

- **Arcsine** $y = \arcsin x$: $\langle -1; 1 \rangle \to \left\langle \frac{\pi}{2}; \frac{\pi}{2} \right\rangle$.
- **Arccosine** $y = \arccos x$: $\langle -1; 1 \rangle \to \langle 0; \pi \rangle$.
- **Arctangent** $y = \arctan x$: $R \to \left(-\frac{\pi}{2}; \frac{\pi}{2} \right)$.
- **Arccotangent** $y = \operatorname{arccot} x$: $R \to (0; \pi)$.

- There are no inverse functions for trigonometric functions because they are not injective. It is necessary to narrow them appropriately.

Elementary Functions II

- Cyclometric functions have the form Maxima `asin(x)`, `acos(x)`, `atan(x)`, `acot(x)`.
- At this point we can mention the function `atan2(x,y)` defined by the relation $\arctan \frac{x}{y}$.

(%i4) `asin(1);asin(1),numer;`
`acos(1);acos(1),numer;`

(%o1) $\frac{\pi}{2}$

(%o2) 1.570796326794897

(%o1) 0

(%o2) 0.0

(%i7) `atan2(2,4);atan(1/2);atan(1/2),numer;`

(%o5) $\mathrm{atan}(\frac{1}{2})$

(%o6) $\mathrm{atan}(\frac{1}{2})$

(%o7) 0.463647609000806

Sum formulas for cyclometric functions.

- $\arcsin x + \arccos x = \frac{\pi}{2}$ for $x \in \langle -1; 1\rangle$.
- $\arctan x + \operatorname{arccot} x = \frac{\pi}{2}$ for $x \in R$.

Elementary Functions II

Hyperbolic functions are:

- **Hyperbolic sine** $y = \sinh x = \frac{e^x - e^{-x}}{2} = \frac{e^{2x} - 1}{2\,e^x}$: $R \to R$.
- **Hyperbolic cosine** $y = \cosh x = \frac{e^x + e^{-x}}{2} = \frac{e^{2x} + 1}{2\,e^x}$: $R \to \langle 1; \infty)$.
- **Hyperbolic tangent** $y = \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$: $R \to (-1; 1)$.
- **Hyperbolic cotangent** $y = \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$: $R - \{0\} \to R - \langle -1; 1 \rangle$.

- Hyperbolic functions have similar properties to trigonometric functions.

Sum formulas for hyperbolic sine and hyperbolic cosine. $x, y \in R$.

- $\sinh(x \pm y) = \sinh x \cdot \cosh y \pm \cosh x \cdot \sinh y$.
- $\cosh(x \pm y) = \cosh x \cdot \cosh y \pm \sinh x \cdot \sinh y$.
- $\sinh x \pm \cosh x = \frac{e^x - e^{-x}}{2} \pm \frac{e^x + e^{-x}}{2} = \pm\, e^{\pm x}$.

- $\sinh^2 x = \frac{\cosh 2x - 1}{2}$.
- $\cosh^2 x = \frac{\cosh 2x + 1}{2}$.
- $\cosh^2 x - \sinh^2 x = 1$.

Elementary Functions II

Hyperbolic functions are:

- **Hyperbolic sine** $y = \sinh x = \frac{e^x - e^{-x}}{2} = \frac{e^{2x} - 1}{2\, e^x}$: $R \to R$.
- **Hyperbolic cosine** $y = \cosh x = \frac{e^x + e^{-x}}{2} = \frac{e^{2x} + 1}{2\, e^x}$: $R \to \langle 1; \infty)$.
- **Hyperbolic tangent** $y = \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$: $R \to (-1; 1)$.
- **Hyperbolic cotangent** $y = \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$: $R - \{0\} \to R - \langle -1; 1 \rangle$.

- Hyperbolic functions have similar properties to trigonometric functions.

Sum formulas for hyperbolic sine and hyperbolic cosine. $x, y \in R$.

- $\sinh(x \pm y) = \sinh x \cdot \cosh y \pm \cosh x \cdot \sinh y$.
- $\cosh(x \pm y) = \cosh x \cdot \cosh y \pm \sinh x \cdot \sinh y$.
- $\sinh x \pm \cosh x = \frac{e^x - e^{-x}}{2} \pm \frac{e^x + e^{-x}}{2} = \pm e^{\pm x}$.
- $\sinh^2 x = \frac{\cosh 2x - 1}{2}$.
- $\cosh^2 x = \frac{\cosh 2x + 1}{2}$.
- $\cosh^2 x - \sinh^2 x = 1$.

Elementary Functions II

Moivre formula. $x \in R$, $n \in N$.

- $(\cosh x \pm \sinh x)^n = \cosh nx \pm \sinh nx$

- Hyperbolic functions are `sinh(x)`, `cosh(x)`, `tanh(x)`, `coth(x)` and to them the inverse hyperbolometric functions are `asinh(x)`, `acosh(x)`, `atanh(x)`, `acoth(x)`.

```
(%i4)  sinh(x);cosh(0);tanh(0);coth(1),numer;
(%o1)  sinh (x)
(%o2)  1
(%o3)  0
(%o4)  1.313035285499331
(%i8)  asinh(x);acosh(1);atanh(0);acoth(1.3),numer;
(%o5)  asinh(x)
(%o6)  0
(%o7)  0
(%o8)  1.01844096363052
```

Elementary Functions II

Hyperbolometric (Inverse hyperbolic) functions are inverses of hyperbolic functions:

- **Inverse hyperbolic sine**

$$y = \operatorname{arsinh} x = \ln\left(x + \sqrt{x^2 + 1}\right): \quad R \to R.$$

- **Inverse hyperbolic cosine**

$$y = \operatorname{arcosh} x = \ln\left(x + \sqrt{x^2 - 1}\right): \quad \langle 1; \infty) \to \langle 0; \infty).$$

- **Inverse hyperbolic tangent**

$$y = \operatorname{artanh} x = \tfrac{1}{2} \ln \tfrac{1+x}{1-x}: \quad (-1; 1) \to R.$$

- **Inverse hyperbolic cotangent**

$$y = \operatorname{arcoth} x = \tfrac{1}{2} \ln \tfrac{x+1}{x-1}: \quad R - \langle -1; 1\rangle \to R - \{0\}.$$

```
(%i3)  ash(x):=log(x+sqrt(x^2+1))$
       a:2$ asinh(a)-ash(a),numer;
(%o3)  0.0
```

Limit of a function

- When investigating a function, it is necessary to characterize its local properties at different intervals and around different important points.
- The function f does not have to be defined at the point around which we investigate it.

Point $a \in R^*$ is an **accumulation point** of the set $A \subset R$,
if for every neighborhood $O(a)$ there exists $x \in O(a) \cap A$, $x \neq a$.

The following definition of limits using sequences is called Heine's.

The function f **has a limit** $b \in R^*$ at the point $a \in R^*$, label $\lim\limits_{x \longrightarrow a} f(x) = b$, if:

- a is the accumulation point of the set $D(f)$.
- For all $\{x_n\}_{n=1}^{\infty} \subset D(f)$, $x_n \neq a$, $\{x_n\}_{n=1}^{\infty} \longrightarrow a$ holds $\{f(x_n)\}_{n=1}^{\infty} \longrightarrow b$.

If $\lim\limits_{x \longrightarrow a} f(x) = b$, then there exists (at least one) $\{x_n\}_{n=1}^{\infty} \longrightarrow a$, $x_n \in D(f) - \{a\}$
and $\lim\limits_{x \longrightarrow a} f(x) = \lim\limits_{n \longrightarrow \infty} f(x_n)$ holds.

Limit of a function

- When investigating a function, it is necessary to characterize its local properties at different intervals and around different important points.
- The function f does not have to be defined at the point around which we investigate it.

Point $a \in R^*$ is an **accumulation point** of the set $A \subset R$,
if for every neighborhood $O(a)$ there exists $x \in O(a) \cap A$, $x \neq a$.

The following definition of limits using sequences is called Heine's.

The function f **has a limit** $b \in R^*$ at the point $a \in R^*$, label $\lim\limits_{x \to a} f(x) = b$, if:

- a is the accumulation point of the set $D(f)$.
- For all $\{x_n\}_{n=1}^{\infty} \subset D(f)$, $x_n \neq a$, $\{x_n\}_{n=1}^{\infty} \longrightarrow a$ holds $\{f(x_n)\}_{n=1}^{\infty} \longrightarrow b$.

If $\lim\limits_{x \to a} f(x) = b$, then there exists (at least one) $\{x_n\}_{n=1}^{\infty} \longrightarrow a$, $x_n \in D(f) - \{a\}$
and $\lim\limits_{x \to a} f(x) = \lim\limits_{n \to \infty} f(x_n)$ holds.

Limit of a function

We can characterize the limit using the neighborhood $O(a)$ and $O(b)$.

The function f **has a limit** $b \in R^*$ at the point $a \in R^*$, label $\lim\limits_{x\to a} f(x) = b$, if:

- a is the accumulation point of the set $D(f)$.
- For every neighborhood $O(b)$ exists a neighborhood $O(a)$ such that for all $x \in O(a)$, $x \neq a$ holds $f(x) \in O(b)$.

$$\lim_{x\to a} f(x) = b. \begin{cases} a \in R^*. & \begin{cases} a \in R. & \text{Limit at eigenpoint } a. \\ a = \pm\infty. & \text{Limit at non-eigenpoint } a. \end{cases} \\ b \in R^*. & \begin{cases} b \in R. & \text{Finite limit.} \\ b = \pm\infty. & \text{Infinite limit.} \end{cases} \end{cases}$$

$\lim\limits_{x\to a} f(x) = b$, where $a \in R^*$, $b \in R$.

$\Rightarrow$
- There exists $O(a)$ in which a function f is bounded.

Limit of a function

We can characterize the limit using the neighborhood $O(a)$ and $O(b)$.

The function f **has a limit** $b \in R^*$ at the point $a \in R^*$, label $\lim\limits_{x \longrightarrow a} f(x) = b$, if:

- a is the accumulation point of the set $D(f)$.
- For every neighborhood $O(b)$ exists a neighborhood $O(a)$ such that for all $x \in O(a)$, $x \neq a$ holds $f(x) \in O(b)$.

$$\lim_{x \longrightarrow a} f(x) = b. \begin{cases} a \in R^*. & \begin{cases} a \in R. & \text{Limit at eigenpoint } a. \\ a = \pm\infty. & \text{Limit at non-eigenpoint } a. \end{cases} \\ b \in R^*. & \begin{cases} b \in R. & \text{Finite limit.} \\ b = \pm\infty. & \text{Infinite limit.} \end{cases} \end{cases}$$

$\lim\limits_{x \longrightarrow a} f(x) = b$, where $a \in R^*$, $b \in R$.

$\Rightarrow$ • There exists $O(a)$ in which a function f is bounded.

Limit of a function

$a \in R^*$ is the accumulation point od sets $D(f)$ and $D(g)$, $O(a)$ is the neighborhood.

$\forall x \in O(a)$, $x \neq a$:
- $f(x) = g(x)$. $\Rightarrow$ $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x)$ if they exist.
- $f(x) \leq g(x)$. $\Rightarrow$ $\lim\limits_{x \to a} f(x) \leq \lim\limits_{x \to a} g(x)$ if they exist.

$\forall x \in O(a)$, $x \neq a$:
- $f(x) < g(x)$. $\Rightarrow$ $\lim\limits_{x \to a} f(x) \leq \lim\limits_{x \to a} g(x)$ if they exist.

Two Policemen and a Drunk Theorem.

$a \in R^*$ is the accumulation point od sets $D(f)$, $D(g)$ and $D(h)$, $O(a)$ is the neighborhood.

$$\left.\begin{array}{l} \bullet\ \forall x \in O(a),\ x \neq a:\ h(x) \leq f(x) \leq g(x). \\ \bullet\ \lim\limits_{x \to a} h(x) = \lim\limits_{x \to a} g(x) = b, \text{ where } b \in R^*. \end{array}\right\} \Rightarrow \bullet\ \text{There exists } \lim\limits_{x \to a} f(x) = b.$$

$$\lim_{x \to \infty} \frac{\sin x}{x} = 0.$$

- ∞ is the accumulation point of the domain $D(f) = R - \{0\}$ of the function $f\colon\ y = \frac{\sin x}{x}$.
- $x > 0$. $\Rightarrow -\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$. $\Rightarrow 0 = -\lim\limits_{x \to \infty} \frac{1}{x} \leq \lim\limits_{x \to \infty} \frac{\sin x}{x} \leq \lim\limits_{x \to \infty} \frac{1}{x} = 0$. $\Rightarrow \lim\limits_{x \to \infty} \frac{\sin x}{x} = 0$.

Limit of a function

The limit of the composite function.

Functions $y = f(x)$, $y = g(x)$, $H(f) \subset D(g)$, $a, b, c \in R^*$, $O(a)$ is a neighborhood.

$$\left.\begin{array}{l} \bullet\ \lim\limits_{x\to a} f(x) = b,\ \lim\limits_{u\to b} g(u) = c. \\ \left.\begin{array}{r} \bullet\ \forall x \in O(a),\ x \neq a\colon\ f(x) \neq b, \\ \text{resp.}\quad \bullet\ g(b) = c. \end{array}\right\} \end{array}\right\} \Rightarrow \bullet\ \lim_{x\to a} g(f(x)) = \lim_{u\to b} g(u) = c.$$

Substitution $u = f(x)$. $\Rightarrow \lim\limits_{x\to a} g(f(x)) = \left[\begin{array}{c} \text{Subst. } u = f(x) \\ x \to a,\ u \to b. \end{array}\right] = \lim\limits_{u\to b} g(u).$

$\lim\limits_{x\to a} f(x) = b$, $\lim\limits_{x\to a} g(x) = c$, $a, b, c \in R^*$, $r \in R$. $\Rightarrow$ (If the expressions make sense.)

- $\lim\limits_{x\to a} |f(x)| = \left|\lim\limits_{x\to a} f(x)\right| = |b|$.
- $\lim\limits_{x\to a} \left[r \cdot f(x)\right] = r \cdot \lim\limits_{x\to a} f(x) = r \cdot b$.
- $\lim\limits_{x\to a} \left[f(x) \circledast g(x)\right] = \lim\limits_{x\to a} f(x) \circledast \lim\limits_{x\to a} g(x) = b \circledast c$, where $\circledast$ is any of $+$, $-$, $\cdot$, resp. $/$.

If any of the expressions do not make sense, it does not mean that the limit does not exist. We have to calculate the limit in another way.

Limit of a function

The limit of the composite function.

Functions $y=f(x)$, $y=g(x)$, $H(f)\subset D(g)$, $a,b,c\in R^*$, $O(a)$ is a neighborhood.

$$\left.\begin{array}{l} \bullet\ \lim\limits_{x\to a} f(x)=b,\ \lim\limits_{u\to b} g(u)=c. \\ \left.\begin{array}{r} \bullet\ \forall x\in O(a),\ x\neq a\!:\ f(x)\neq b, \\ \text{resp.}\quad \bullet\ g(b)=c. \end{array}\right\} \end{array}\right\} \Rightarrow \ \bullet\ \lim_{x\to a} g(f(x))=\lim_{u\to b} g(u)=c.$$

Substitution $u=f(x)$. $\Rightarrow \lim\limits_{x\to a} g(f(x))=\left[\begin{array}{c}\text{Subst. } u=f(x)\\ x\to a,\ u\to b.\end{array}\right]=\lim\limits_{u\to b} g(u).$

$\lim\limits_{x\to a} f(x)=b$, $\lim\limits_{x\to a} g(x)=c$, $a,b,c\in R^*$, $r\in R$. $\Rightarrow$ (If the expressions make sense.)

- $\lim\limits_{x\to a} |f(x)|=\left|\lim\limits_{x\to a} f(x)\right|=|b|.$
- $\lim\limits_{x\to a} \left[r\cdot f(x)\right]=r\cdot\lim\limits_{x\to a} f(x)=r\cdot b.$
- $\lim\limits_{x\to a} \left[f(x)\circledast g(x)\right]=\lim\limits_{x\to a} f(x)\circledast\lim\limits_{x\to a} g(x)=b\circledast c,$ where $\circledast$ is any of $+$, $-$, $\cdot$, resp. $/$.

If any of the expressions do not make sense, it does not mean that the limit does not exist. We have to calculate the limit in another way.

Limit of a function

A function $y = f(x)$, $x \in D(f)$, a point $a \in R$.

- $f^{-}(x) = f(x)|_{D(f)\cap(-\infty;a)} = f(x)|_{\{x\in D(f),\ x<a\}}$ Narrowing the function f to the left.
- $f^{+}(x) = f(x)|_{D(f)\cap(a;\infty)} = f(x)|_{\{x\in D(f),\ a<x\}}$ Narrowing the function f to the right.

- $\lim\limits_{x\to a^{-}} f(x) = \lim\limits_{x\to a} f^{-}(x)$ Left limit (from left).
- $\lim\limits_{x\to a^{+}} f(x) = \lim\limits_{x\to a} f^{+}(x)$ Right limit (from right).

$\Big\}$ One-sided limits of the function f at the point a.

- $\lim\limits_{x\to a} f(x)$ (Two-sided) limit of the function f at the point a.

```
(%i3) limit(1/x,x,0,minus);limit(1/x,x,0,plus);
      limit(1/x,x,0);
(%o1) −∞
(%o2) ∞
(%o3) infinity        /* Complex inf */
```

If $a \in R$, $b \in R^{*}$, then holds: • $\lim\limits_{x\to a} f(x) = b.$ $\Leftrightarrow$ • $\lim\limits_{x\to a^{-}} f(x) = \lim\limits_{x\to a^{+}} f(x) = b.$

Limit of a function

A function $y = f(x)$, $x \in D(f)$, a point $a \in R$.

- $f^{-}(x) = f(x)|_{D(f)\cap(-\infty;a)} = f(x)|_{\{x\in D(f),\ x<a\}}$ Narrowing the function f to the left.
- $f^{+}(x) = f(x)|_{D(f)\cap(a;\infty)} = f(x)|_{\{x\in D(f),\ a<x\}}$ Narrowing the function f to the right.

- $\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a} f^{-}(x)$ Left limit (from left).
- $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a} f^{+}(x)$ Right limit (from right).

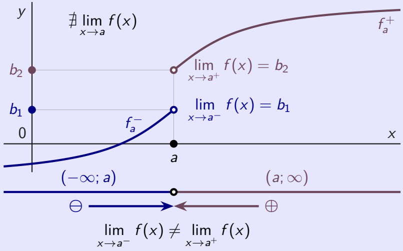
} One-sided limits of the function f at the point a.

- $\lim\limits_{x\to a} f(x)$ (Two-sided) limit of the function f at the point a.

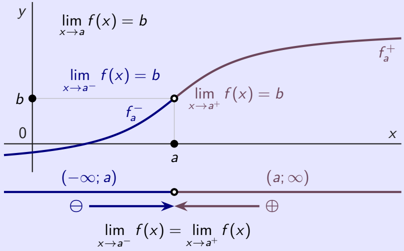
```
(%i3) limit(1/x,x,0,minus);limit(1/x,x,0,plus);
      limit(1/x,x,0);
(%o1) -∞
(%o2) ∞
(%o3) infinity      /* Complex inf */
```

If $a \in R$, $b \in R^*$, then holds: • $\lim\limits_{x\to a} f(x) = b$. $\Leftrightarrow$ • $\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} f(x) = b$.

Limit of a function



One-sided limits

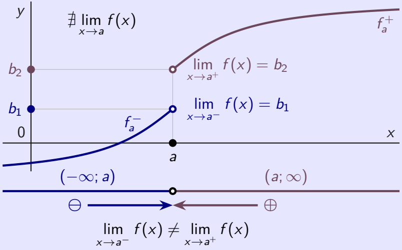


Two-sided limit

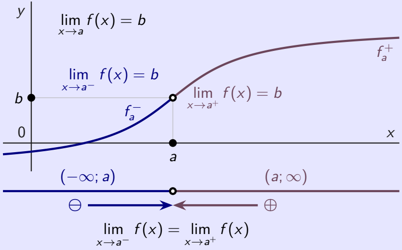
Important limits. $a, b \in R,\ a > 0.$

- $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1.$
- $\lim\limits_{x\to \infty} \left(1 + \frac{b}{x}\right)^x = e^b.$
- $\lim\limits_{x\to \infty} \left(1 + \frac{1}{x}\right)^x = e.$
- $\lim\limits_{x\to 0} \frac{\arcsin x}{x} = 1.$
- $\lim\limits_{x\to 0} \sqrt[x]{1 + bx} = e^b.$
- $\lim\limits_{x\to 0} \sqrt[x]{1 + x} = e.$
- $\lim\limits_{x\to \infty} \sqrt[x]{x} = 1.$
- $\lim\limits_{x\to \infty} x\left(\sqrt[x]{a} - 1\right) = \ln a.$
- $\lim\limits_{x\to \infty} x\left(\sqrt[x]{e} - 1\right) = \ln e = 1.$
- $\lim\limits_{x\to \infty} \sqrt[x]{a} = 1.$
- $\lim\limits_{x\to 0} \frac{a^x - 1}{x} = \ln a.$
- $\lim\limits_{x\to 0} \frac{e^x - 1}{x} = \ln e = 1.$

Limit of a function



One-sided limits



Two-sided limit

Important limits.

$a, b \in R,\ a > 0.$

- $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1.$
- $\lim\limits_{x\to 0} \frac{\arcsin x}{x} = 1.$
- $\lim\limits_{x\to \infty} \sqrt[x]{x} = 1.$
- $\lim\limits_{x\to \infty} \sqrt[x]{a} = 1.$
- $\lim\limits_{x\to \infty} \left(1 + \frac{b}{x}\right)^x = \mathrm{e}^b.$
- $\lim\limits_{x\to 0} \sqrt[x]{1 + bx} = \mathrm{e}^b.$
- $\lim\limits_{x\to \infty} x \left(\sqrt[x]{a} - 1\right) = \ln a.$
- $\lim\limits_{x\to 0} \frac{a^x - 1}{x} = \ln a.$
- $\lim\limits_{x\to \infty} \left(1 + \frac{1}{x}\right)^x = \mathrm{e}.$
- $\lim\limits_{x\to 0} \sqrt[x]{1 + x} = \mathrm{e}.$
- $\lim\limits_{x\to \infty} x \left(\sqrt[x]{\mathrm{e}} - 1\right) = \ln \mathrm{e} = 1.$
- $\lim\limits_{x\to 0} \frac{\mathrm{e}^x - 1}{x} = \ln \mathrm{e} = 1.$

Limit of a function

When investigating the function f , it is important to examine its properties at non-eigenpoints:

- For $x \to \pm\infty$.
- In the neighborhood $O(a)$ of the point $a \in R$,
  for which holds $\lim\limits_{x \to a^-} f(x) = \pm\infty$ or $\lim\limits_{x \to a^+} f(x) = \pm\infty$.

A function $y = f(x)$, $x \in D(f)$, a point $a \in R$.

- The line $x = a$ is called **asymptote without slope** (**vertical**) of the graph f,
  if $\lim\limits_{x \to a^-} f(x) = \pm\infty$ or $\lim\limits_{x \to a^+} f(x) = \pm\infty$ (at least one of the limits is infinite).
- The line $y = kx + q$ is called the **asymptote with slope** of the graph f,
  if $\lim\limits_{x \to -\infty} [f(x) - (kx + q)] = 0$ or $\lim\limits_{x \to \infty} [f(x) - (kx + q)] = 0$.

Specially the asymptote $y = q$ is called **horizontal asymptote**,

i.e. $k = 0$ and $\lim\limits_{x \to -\infty} f(x) = q$ or $\lim\limits_{x \to \infty} f(x) = q$.

Limit of a function

When investigating the function f , it is important to examine its properties at non-eigenpoints:

- For $x \to \pm\infty$.
- In the neighborhood $O(a)$ of the point $a \in R$,
  for which holds $\lim\limits_{x\to a^-} f(x) = \pm\infty$ or $\lim\limits_{x\to a^+} f(x) = \pm\infty$.

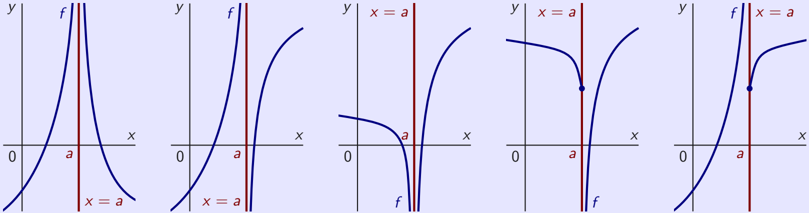
A function $y = f(x)$, $x \in D(f)$, a point $a \in R$.

- The line $x = a$ is called **asymptote without slope** (**vertical**) of the graph f,
  if $\lim\limits_{x\to a^-} f(x) = \pm\infty$ or $\lim\limits_{x\to a^+} f(x) = \pm\infty$ (at least one of the limits is infinite).
- The line $y = kx + q$ is called the **asymptote with slope** of the graph f,
  if $\lim\limits_{x\to -\infty} [f(x) - (kx+q)] = 0$ or $\lim\limits_{x\to \infty} [f(x) - (kx+q)] = 0$.

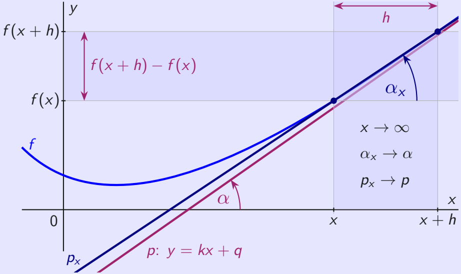
Specially the asymptote $y = q$ is called **horizontal asymptote**,

i.e. $k = 0$ and $\lim\limits_{x\to -\infty} f(x) = q$ or $\lim\limits_{x\to \infty} f(x) = q$.

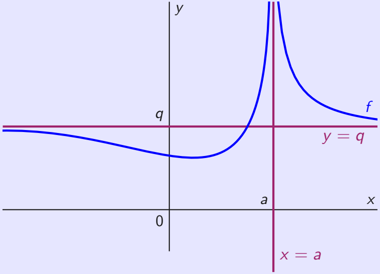
Limit of a function



Examples of asymptotes without slope.



Asymptote with slope α.



Vertical asymptote $y = q$.
Horizontal asymptote $x = a$.

Limit of a function

The functions $y = f(x)$, $x \in D(f)$ and a domain $D(f)$ is an unbounded set.

- The line $y = kx + q$ is an asymptote with the slope of the graph f.

$$\Leftrightarrow \quad \bullet \text{ Exist } \lim_{x\to\pm\infty} \frac{f(x)}{x} = k,\ \lim_{x\to\pm\infty} [f(x) - kx] = q,\ k, q \in R.$$

$$\lim_{x\to\infty} \frac{f(x)-(kx+q)}{x} = \lim_{x\to\infty} \left[\frac{f(x)}{x} - k - \frac{q}{x}\right] = 0. \qquad \Rightarrow \lim_{x\to\infty} \frac{f(x)}{x} = k.$$

$$\lim_{x\to\infty} [f(x) - (kx + q)] = \lim_{x\to\infty} [(f(x) - kx) - q] = 0. \qquad \Rightarrow \lim_{x\to\infty} [f(x) - kx] = q.$$

A function $f(x) = \frac{2x^2+x+1}{8x}$, $x \in R$.

- $k = \lim\limits_{x\to\pm\infty} \frac{f(x)}{x} = \lim\limits_{x\to\pm\infty} \frac{2x^2+x+1}{8x^2} = \lim\limits_{x\to\pm\infty} \frac{x^2(2+\frac{1}{x}+\frac{1}{x^2})}{8x^2} = \lim\limits_{x\to\pm\infty} \frac{2+\frac{1}{x}+\frac{1}{x^2}}{8} = \frac{2+0+0}{8} = \frac{1}{4}.$
- $q = \lim\limits_{x\to\pm\infty} [f(x) - kx] = \lim\limits_{x\to\pm\infty} \left[\frac{2x^2+x+1}{8x} - \frac{x}{4}\right]$

$$= \lim_{x\to\pm\infty} \left[\frac{2x^2+x+1}{8x} - \frac{2x^2}{8x}\right] = \lim_{x\to\pm\infty} \frac{x+1}{8x} = \lim_{x\to\pm\infty} \left[\frac{1}{8} + \frac{1}{8x}\right] = \frac{1}{8}.$$

- The line $y = \frac{x}{4} + \frac{1}{8}$ is an asymptote with the slope $\frac{1}{4}$.

Limit of a function

The functions $y = f(x)$, $x \in D(f)$ and a domain $D(f)$ is an unbounded set.

- The line $y = kx + q$ is an asymptote with the slope of the graph f.

$$\Leftrightarrow \quad \bullet \text{ Exist } \lim_{x\to\pm\infty} \frac{f(x)}{x} = k,\ \lim_{x\to\pm\infty} [f(x) - kx] = q,\ k, q \in R.$$

$$\lim_{x\to\infty} \frac{f(x)-(kx+q)}{x} = \lim_{x\to\infty} \left[\frac{f(x)}{x} - k - \frac{q}{x}\right] = 0. \qquad \Rightarrow \lim_{x\to\infty} \frac{f(x)}{x} = k.$$

$$\lim_{x\to\infty} [f(x) - (kx+q)] = \lim_{x\to\infty} [(f(x) - kx) - q] = 0. \qquad \Rightarrow \lim_{x\to\infty} [f(x) - kx] = q.$$

A function $f(x) = \frac{2x^2+x+1}{8x}$, $x \in R$.

- $k = \lim\limits_{x\to\pm\infty} \frac{f(x)}{x} = \lim\limits_{x\to\pm\infty} \frac{2x^2+x+1}{8x^2} = \lim\limits_{x\to\pm\infty} \frac{x^2(2+\frac{1}{x}+\frac{1}{x^2})}{8x^2} = \lim\limits_{x\to\pm\infty} \frac{2+\frac{1}{x}+\frac{1}{x^2}}{8} = \frac{2+0+0}{8} = \frac{1}{4}.$
- $q = \lim\limits_{x\to\pm\infty} [f(x) - kx] = \lim\limits_{x\to\pm\infty} \left[\frac{2x^2+x+1}{8x} - \frac{x}{4}\right]$

$$= \lim_{x\to\pm\infty} \left[\frac{2x^2+x+1}{8x} - \frac{2x^2}{8x}\right] = \lim_{x\to\pm\infty} \frac{x+1}{8x} = \lim_{x\to\pm\infty} \left[\frac{1}{8} + \frac{1}{8x}\right] = \frac{1}{8}.$$

- The line $y = \frac{x}{4} + \frac{1}{8}$ is an asymptote with the slope $\frac{1}{4}$.

Continuity of function

- The concept of the limit of the function f at the point a is closely related to the concept of the continuity of f at the point a.
- Continuity is also a local matter in some neighborhood of $O(a)$.

The following definition of continuity at the point $a \in D(f)$ using sequences is called Heine.

The function f **is continuous at the point** $a \in D(f)$, if:

- For all $\{x_n\}_{n=1}^{\infty} \subset D(f)$, $\{x_n\}_{n=1}^{\infty} \longrightarrow a$ holds $\{f(x_n)\}_{n=1}^{\infty} \longrightarrow f(a)$.

- If $a \in D(f)$ is an isolated point, then the function f is continuous at point a.

  (Then there exists a single $\{x_n\}_{n=1}^{\infty} = \{a\}_{n=1}^{\infty} \longrightarrow a$.)

We can characterize the continuity using the neighborhood $O(a)$ and $O(f(a))$.

The function f **is continuous at the point** $a \in D(f)$, if:

- For every neighborhood $O(f(a))$ exists a neighborhood $O(a)$ such that for all $x \in O(a)$ holds $f(x) \in O(f(a))$, i.e. $f(O(a)) \subset O(f(a))$.

Continuity of function

- The concept of the limit of the function f at the point a is closely related to the concept of the continuity of f at the point a.
- Continuity is also a local matter in some neighborhood of $O(a)$.

The following definition of continuity at the point $a \in D(f)$ using sequences is called Heine.

The function f **is continuous at the point** $a \in D(f)$, if:

- For all $\{x_n\}_{n=1}^{\infty} \subset D(f)$, $\{x_n\}_{n=1}^{\infty} \longrightarrow a$ holds $\{f(x_n)\}_{n=1}^{\infty} \longrightarrow f(a)$.

- If $a \in D(f)$ is an isolated point, then the function f is continuous at point a.

(Then there exists a single $\{x_n\}_{n=1}^{\infty} = \{a\}_{n=1}^{\infty} \longrightarrow a$.)

We can characterize the continuity using the neighborhood $O(a)$ and $O(f(a))$.

The function f **is continuous at the point** $a \in D(f)$, if:

- For every neighborhood $O(f(a))$ exists a neighborhood $O(a)$ such that
  for all $x \in O(a)$ holds $f(x) \in O(f(a))$, i.e. $f(O(a)) \subset O(f(a))$.

Continuity of function

If $a \in D(f)$ is an accumulation point,
then the definition of continuity coincides with the definition of limit.

A function $y = f(x)$, $x \in D(f)$, $a \in D(f)$ is an accumulation point $D(f)$.

- The function f is continuous at point a. ⇔ • $\lim\limits_{x \to a} f(x) = f(a)$.

The functions f, g are continuous at the point $a \in D(f) \cap D(g)$, $r \in R$.

$\Rightarrow$ • $|f|$, • $f \pm g$, • rf, • fg, • $\frac{f}{g}$ for $g(a) \neq 0$ are continuous at point a.

Continuity of a composite function.

- A function f is continuous at the point $a \in D(f)$.
- A function g is continuous at the point $b = f(a) \in D(g)$.
- $H(f) \subset D(g)$.

$\Rightarrow$ • The function $F = g(f)$ is continuous at point a.

Continuity of function

A function $y = f(x)$, $x \in D(f)$, a point $a \in D(f)$.

- $f_a^-(x) = f(x)|_{D(f) \cap (-\infty; a\rangle} = f(x)|_{\{x \in D(f),\ x \leq a\}}$ Narrowing the function f to the left.
- $f_a^+(x) = f(x)|_{D(f) \cap \langle a; \infty)} = f(x)|_{\{x \in D(f),\ a \leq x\}}$ Narrowing the function f to the right.

- $f_a^-(x)$ continuous at point a Continuity from left.
- $f_a^+(x)$ continuous at point a Continuity from right.

} One-sided continuity of the function f at the point a.

- $f(x)$ continuous at point a (Two-sided) continuity of the function f at the point a.

A function $y = f(x)$, $x \in D(f)$, a point $a \in D(f)$, a set $A \subset D(f)$.

- A function f is continuous at the point $a \in D(f)$.
  $\Rightarrow$ • There exists $O(a)$ in which f is bounded.
- A function f is continuous on the set $A \subset D(f)$.
  $\Rightarrow$ • The function f need not be bounded on A.

A function f is called **continuous on the set** $A \subset D(f)$, if it is continuous at every point $a \in A$.

Continuity of function

A function $y = f(x)$, $x \in D(f)$, a point $a \in D(f)$.

- $f_a^-(x) = f(x)|_{D(f)\cap(-\infty;a\rangle} = f(x)|_{\{x\in D(f),\ x\leq a\}}$ Narrowing the function f to the left.
- $f_a^+(x) = f(x)|_{D(f)\cap\langle a;\infty)} = f(x)|_{\{x\in D(f),\ a\leq x\}}$ Narrowing the function f to the right.

- $f_a^-(x)$ continuous at point a Continuity from left.
- $f_a^+(x)$ continuous at point a Continuity from right.

} One-sided continuity of the function f at the point a.

- $f(x)$ continuous at point a (Two-sided) continuity of the function f at the point a.

A function $y = f(x)$, $x \in D(f)$, a point $a \in D(f)$, a set $A \subset D(f)$.

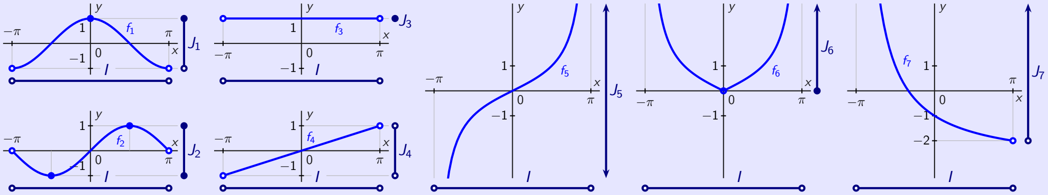
- A function f is continuous at the point $a \in D(f)$.
  $\Rightarrow$ • There exists $O(a)$ in which f is bounded.
- A function f is continuous on the set $A \subset D(f)$.
  $\Rightarrow$ • The function f need not be bounded on A.

A function f is called **continuous on the set** $A \subset D(f)$, if it is continuous at every point $a \in A$.

Continuity of function

If A function f is continuous on the interval $I \subset R$, then the set $f(I)$ is an interval.

- $I = \langle a; b \rangle$ is a closed interval. $\Rightarrow$ • The set $f(I)$ is a closed interval.
- I is not a closed interval. $\Rightarrow$ • The set $f(I)$ can be an interval of different types.



- $f_1(x) = \cos x$: $(-\pi; \pi) \to J_1 = (-1; 1\rangle$.
- $f_2(x) = \sin x$: $(-\pi; \pi) \to J_2 = \langle -1; 1\rangle$.
- $f_3(x) = 1$: $(-\pi; \pi) \to J_3 = \{1\}$.
- $f_4(x) = \frac{x}{\pi}$: $(-\pi; \pi) \to J_4 = (-1; 1)$.
- $f_5(x) = \tan \frac{x}{2}$: $(-\pi; \pi) \to J_5 = (-\infty; \infty)$.
- $f_6(x) = \left|\tan \frac{x}{2}\right|$: $(-\pi; \pi) \to J_6 = \langle 0; \infty)$.
- $f_7(x) = -\frac{3x+\pi}{x+\pi}$: $(-\pi; \pi) \to J_7 = (-2; \infty)$.

Continuity of function

A function f can be discontinuous only at an accumulation point $a\in R$ (point of discontinuity).

- Removable discontinuity

  There exists $\lim\limits_{x\to a} f(x) = b\in R,\ b\neq f(a)$.

- Non-removable discontinuity of the type I.

  $$\text{There exist } \left.\begin{array}{r} \lim\limits_{x\to a^-} f(x) = b^-\in R \\ \text{and } \lim\limits_{x\to a^+} f(x) = b^+\in R \end{array}\right\} b^-\neq b^+.$$

  The difference $c = b^+ - b^-$ is called
  the jump of the function f at the point a.

- Non-removable discontinuity of the type II.

  $$\text{At least one of } \left.\begin{array}{r} \lim\limits_{x\to a^-} f(x) \\ \text{or } \lim\limits_{x\to a^+} f(x) \end{array}\right\} \begin{array}{l}\text{does not exist} \\ \text{or is infinite.}\end{array}$$

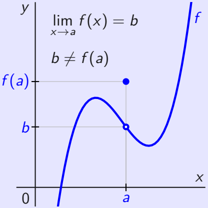
  Asymptotic discontinuity,
  if one of the one-sided limits is infinite.

The function f is discontinuous
at the point $a\in R$.

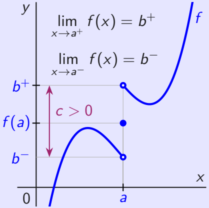
Value $f(a)$ may, but may not exist.

Continuity of function

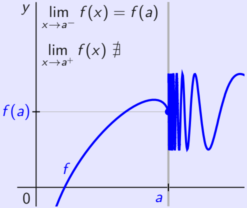
- Discontinuity of function f at point a.



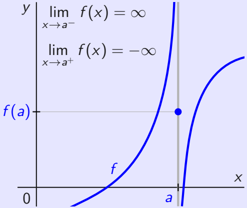
Removable discontinuity.



Non-removable discontinuity of type I.



Non-removable discontinuity of type II.



Non-removable discontinuity of type II (asymptotic discontinuity).

Cauchy's zero point theorem.

- A function f is continuous on $\langle a; b\rangle$.
- $f(a) \cdot f(b) < 0$.

$\Rightarrow$

- There exists $c \in (a; b)$ such that $f(c) = 0$.

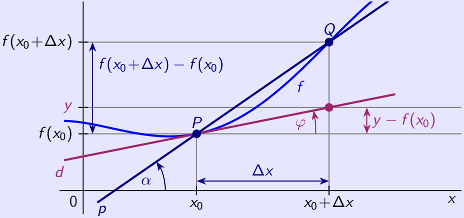
Differential calculus

Mathematical Analysis supported by wxMaxima

Derivative of a real function

The function $y = f(x)$, $x \in D(f)$ is continuous.

- The points $P = [x_0; f(x_0)]$, $Q = [x_0+\Delta x; f(x_0+\Delta x)]$ lie on graph f.
- The line PQ has the slope $\tan\alpha = \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$.
- The tangent line to f at point P has the form d_P: $y - f(x_0) = \tan\varphi \cdot \Delta x$, where $\tan\varphi = \frac{y-f(x_0)}{\Delta x}$ is its slope.



- $Q \to P$. $\Rightarrow$
- $\Delta x \to 0$.
  - $x_0 + \Delta x \to x_0$,
  - $f(x_0 + \Delta x) \to f(x_0)$.
  - $\alpha \to \varphi$,
  - $\tan\alpha \to \tan\varphi$.
  - $PQ \to d_P$ (the line PQ to the tangent line).

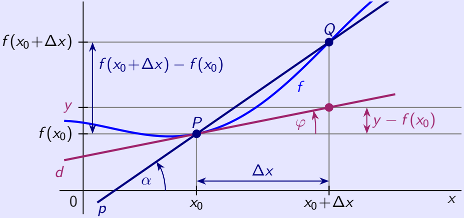
- The tangent line has the slope $\tan\varphi = \lim\limits_{\alpha\to\varphi} \tan\alpha = \lim\limits_{\Delta x\to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$.

Geometric meaning of the derivative of a function at a point. – The slope of the tangent line.

Derivative of a real function

The function $y = f(x)$, $x \in D(f)$ is continuous.

- The points $P = [x_0; f(x_0)]$, $Q = [x_0 + \Delta x; f(x_0 + \Delta x)]$ lie on graph f.
- The line PQ has the slope $\tan\alpha = \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$.
- The tangent line to f at point P has the form d_P: $y - f(x_0) = \tan\varphi \cdot \Delta x$, where $\tan\varphi = \frac{y - f(x_0)}{\Delta x}$ is its slope.



- $Q \to P$. $\Rightarrow$
- $\Delta x \to 0$.
  - $x_0 + \Delta x \to x_0$,
  - $f(x_0 + \Delta x) \to f(x_0)$.
  - $\alpha \to \varphi$,
  - $\tan\alpha \to \tan\varphi$.
  - $PQ \to d_P$ (the line PQ to the tangent line).

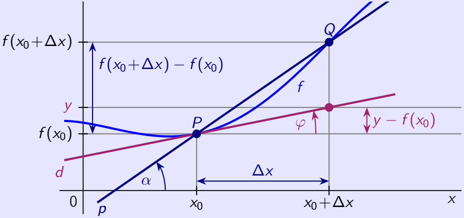
- The tangent line has the slope $\tan\varphi = \lim\limits_{\alpha\to\varphi} \tan\alpha = \lim\limits_{\Delta x\to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$.

Geometric meaning of the derivative of a function at a point. – The slope of the tangent line.

Derivative of a real function

The function $y = f(x)$, $x \in D(f)$ is continuous.

- The points $P = [x_0; f(x_0)]$, $Q = [x_0 + \Delta x; f(x_0 + \Delta x)]$ lie on graph f.
- The line PQ has the slope $\tan\alpha = \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$.
- The tangent line to f at point P has the form d_P: $y - f(x_0) = \tan\varphi \cdot \Delta x$,
  where $\tan\varphi = \frac{y - f(x_0)}{\Delta x}$ is its slope.



- $Q \to P$. $\Rightarrow$ • $\Delta x \to 0$.
  - $x_0 + \Delta x \to x_0$, • $f(x_0 + \Delta x) \to f(x_0)$.
  - $\alpha \to \varphi$, • $\tan\alpha \to \tan\varphi$.
  - $PQ \to d_P$ (the line PQ to the tangent line).

- The tangent line has the slope $\tan\varphi = \lim\limits_{\alpha\to\varphi} \tan\alpha = \lim\limits_{\Delta x\to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$.

Geometric meaning of the derivative of a function at a point. – The slope of the tangent line.

Derivative of a real function

A function $y=f(x)$, $x\in D(f)$ has a **derivative at the point** $x_0\in D(f)$,

label $f'(x_0)$, resp. $y'(x_0)$ or $f'(x_0)=\frac{\mathrm{d}f(x_0)}{\mathrm{d}x}$, resp. $y'(x_0)=\frac{\mathrm{d}y(x_0)}{\mathrm{d}x}$ using differentials,

if it exists • $\lim\limits_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=\begin{bmatrix}\text{Subst. } h=x-x_0\\ x\to x_0, \quad h\to 0\end{bmatrix}=\lim\limits_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}=f'(x_0).$

$$\left.\begin{array}{ll}\bullet\ f'(x_0)\in R. & \text{Eigen (finite)}\\ \bullet\ f'(x_0)=\infty \text{ or } f'(x_0)=-\infty. & \text{Non-eigen (infinite)}\end{array}\right\}\ \text{derivative of the } f \text{ at the point } x_0.$$

A function $y=f(x)$, $x\in D(f)$, point $x_o\in D(f)$.

- There exists $f'(x_0)\in R$ (finite). ⇒ • f is continuous at the point x_0.

The continuity of the function f at the point x_0 does not guarantee the existence of $f'(x_0)$.

The function $f\colon\ y=|x|$ is continuous at the point $x_0=0$.

- But, there does not exist $f'(0)=\lim\limits_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0}\frac{|x|-|0|}{x-0}=\lim\limits_{x\to 0}\frac{|x|}{x}=\begin{cases}\lim\limits_{x\to 0^+}\frac{x}{x}=1.\\ \lim\limits_{x\to 0^+}\frac{-x}{x}=-1.\end{cases}$

Derivative of a real function

A function $y=f(x)$, $x\in D(f)$ has a **derivative at the point** $x_0\in D(f)$,
label $f'(x_0)$, resp. $y'(x_0)$ or $f'(x_0)=\frac{\mathrm{d}f(x_0)}{\mathrm{d}x}$, resp. $y'(x_0)=\frac{\mathrm{d}y(x_0)}{\mathrm{d}x}$ using differentials,

if it exists • $\lim\limits_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=\left[\begin{array}{ll}\text{Subst. } & h=x-x_0\\ x\to x_0, & h\to 0\end{array}\right]=\lim\limits_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}=f'(x_0).$

$$\left.\begin{array}{ll}\bullet\ f'(x_0)\in R. & \text{Eigen (finite)}\\ \bullet\ f'(x_0)=\infty \text{ or } f'(x_0)=-\infty. & \text{Non-eigen (infinite)}\end{array}\right\}\ \text{derivative of the } f \text{ at the point } x_0.$$

A function $y=f(x)$, $x\in D(f)$, point $x_o\in D(f)$.

- There exists $f'(x_0)\in R$ (finite). $\Rightarrow$ • f is continuous at the point x_0.

The continuity of the function f at the point x_0 does not guarantee the existence of $f'(x_0)$.

The function $f\colon\ y=|x|$ is continuous at the point $x_0=0$.

- But, there does not exist $f'(0)=\lim\limits_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0}\frac{|x|-|0|}{x-0}=\lim\limits_{x\to 0}\frac{|x|}{x}=\begin{cases}\lim\limits_{x\to 0^+}\frac{x}{x}=1.\\ \lim\limits_{x\to 0^+}\frac{-x}{x}=-1.\end{cases}$

Derivative of a real function

$f'(x_0)$ represents geometrically the slope of the tangent line to the graph f at the point x_0.

- $f'(x_0) \in R$. Tangent line d: $y = f(x_0) + f'(x_0)(x - x_0)$ with slope $f'(x_0)$.
- $f'(x_0) = \pm\infty$ and f is continuous at the point x_0.
  Tangent line d: $x = x_0$ without slope (vertical).

We calculate the derivative of the function $f(x) = \ln\left(x + \sqrt{x^2+1}\right)$.

(%i1) `f(x):=log(x+sqrt(x^2+1));`

(%o1) $f(x) := \log\left(x + \sqrt{x^2+1}\right)$

(%i3) `f_1(x):=diff(f(x),x);f_1(x);`

(%o2) $f\_1(x) := \frac{d}{dx} f(x)$

(%o3) $\frac{\frac{x}{\sqrt{x^2+1}}+1}{\sqrt{x^2+1}+x}$

(%i4) `ratsimp(f1(x));`

(%o4) $\frac{\sqrt{x^2+1}+x}{x\sqrt{x^2+1}+x^2+1}$

Derivative of a real function

$f'(x_0)$ represents geometrically the slope of the tangent line to the graph f at the point x_0.

- $f'(x_0) \in R$. Tangent line d: $y = f(x_0) + f'(x_0)(x - x_0)$ with slope $f'(x_0)$.
- $f'(x_0) = \pm\infty$ and f is continuous at the point x_0.
  Tangent line d: $x = x_0$ without slope (vertical).

We calculate the derivative of the function $f(x) = \ln\left(x + \sqrt{x^2+1}\right)$.

(%i1) `f(x):=log(x+sqrt(x^2+1));`

(%o1) $f(x) := \log\left(x + \sqrt{x^2+1}\right)$

(%i3) `f_1(x):=diff(f(x),x);f_1(x);`

(%o2) $f\_1(x) := \frac{d}{dx} f(x)$

(%o3) $\frac{\frac{x}{\sqrt{x^2+1}}+1}{\sqrt{x^2+1}+x}$

(%i4) `ratsimp(f1(x));`

(%o4) $\frac{\sqrt{x^2+1}+x}{x\sqrt{x^2+1}+x^2+1}$

Derivative of a real function

A function $y = f(x)$, $x \in D(f)$, a point $x_0 \in D(f)$.

$$\left.\begin{array}{ll} \bullet\ f'_-(x_0) = \lim\limits_{x\to x_0^-} \frac{f(x)-f(x_0)}{x-x_0} & \text{Derivative from left.} \\ \bullet\ f'_+(x_0) = \lim\limits_{x\to x_0^+} \frac{f(x)-f(x_0)}{x-x_0} & \text{Derivative from right.} \end{array}\right\} \begin{array}{l}\text{One-sided derivatives} \\ \quad \text{of the function } f \text{ at the point } x_0.\end{array}$$

- $f'(x_0) = \lim\limits_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ (Two-sided) derivative of the function f at the point x_0.

A function $y = f(x)$, $x \in D(f)$, a set $A \subset \{x_0 \in D(f), f'(x_0) \text{ is finite}\}$, $A \neq \emptyset$.

- Then $y = f'(x)$, $x \in A$ is a function
  and is called the **derivative** of the function f on the set A, label $f' = \frac{\mathrm{d}f}{\mathrm{d}x}$, resp. $y' = \frac{\mathrm{d}y}{\mathrm{d}x}$.

A function $y = f(x)$, $x \in D(f)$, a set $A \subset D(f)$.

- $\forall x_0 \in A$: $f'(x_0) \in R$ (finite derivative). $\Rightarrow$ • The function f is continuous on the set A.

The exponencial function f: $y = \mathrm{e}^x$, $x \in R$.

- $[\mathrm{e}^x]' = \lim\limits_{h\to0} \frac{\mathrm{e}^{x+h}-\mathrm{e}^x}{h} = \lim\limits_{h\to0} \frac{\mathrm{e}^x\cdot(\mathrm{e}^h-1)}{h} = \mathrm{e}^x \cdot \lim\limits_{h\to0} \frac{\mathrm{e}^h-1}{h} = \mathrm{e}^x \cdot 1 = \mathrm{e}^x$ for all $x \in R$.

Derivative of a real function

A function $y = f(x)$, $x \in D(f)$, a point $x_0 \in D(f)$.

- $f'_-(x_0) = \lim\limits_{x \to x_0^-} \frac{f(x)-f(x_0)}{x-x_0}$ Derivative from left.
- $f'_+(x_0) = \lim\limits_{x \to x_0^+} \frac{f(x)-f(x_0)}{x-x_0}$ Derivative from right.

$\Big\}$ One-sided derivatives of the function f at the point x_0.

- $f'(x_0) = \lim\limits_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ (Two-sided) derivative of the function f at the point x_0.

A function $y = f(x)$, $x \in D(f)$, a set $A \subset \{x_0 \in D(f), f'(x_0) \text{ is finite}\}$, $A \neq \emptyset$.

- Then $y = f'(x)$, $x \in A$ is a function
  and is called the **derivative** of the function f on the set A, label $f' = \frac{\mathrm{d}f}{\mathrm{d}x}$, resp. $y' = \frac{\mathrm{d}y}{\mathrm{d}x}$.

A function $y = f(x)$, $x \in D(f)$, a set $A \subset D(f)$.

- $\forall x_0 \in A$: $f'(x_0) \in R$ (finite derivative). $\Rightarrow$
- The function f is continuous on the set A.

The exponencial function f: $y = \mathrm{e}^x$, $x \in R$.

- $[\mathrm{e}^x]' = \lim\limits_{h \to 0} \frac{\mathrm{e}^{x+h} - \mathrm{e}^x}{h} = \lim\limits_{h \to 0} \frac{\mathrm{e}^x \cdot (\mathrm{e}^h - 1)}{h} = \mathrm{e}^x \cdot \lim\limits_{h \to 0} \frac{\mathrm{e}^h - 1}{h} = \mathrm{e}^x \cdot 1 = \mathrm{e}^x$ for all $x \in R$.

Derivative of a real function

Derivatives of basic elementary functions.

- $[c]' = 0$ for $x \in R$, $c \in R$.
- $[x^n]' = nx^{n-1}$ for $x \in R$, $n \in N$.
- $[e^x]' = e^x$ for $x \in R$.
- $[\ln x]' = \frac{1}{x}$ for $x > 0$.
- $[\sin x]' = \cos x$ for $x \in R$.
- $[\tan x]' = \frac{1}{\cos^2 x}$ for $x \neq \frac{(2k+1)\pi}{2}$, $k \in Z$.
- $[\arcsin x]' = \frac{1}{\sqrt{1-x^2}}$ for $x \in (-1; 1)$.
- $[\operatorname{arctan} x]' = \frac{1}{1+x^2}$ for $x \in R$.
- $[\sinh x]' = \cosh x$ for $x \in R$.
- $[\tanh x]' = \frac{1}{\cosh^2 x}$ for $x \in R$.
- $[\operatorname{arsinh} x]' = \frac{1}{\sqrt{x^2+1}}$ for $x \in R$.
- $[\operatorname{artanh} x]' = \frac{1}{1-x^2}$ for $x \in (-1; 1)$.
- $[x]' = 1$ for $x \in R$.
- $[x^a]' = ax^{a-1}$ for $x > 0$, $a \in R$.
- $[a^x]' = a^x \ln a$ for $x \in R$, $a > 0$.
- $[\log_a x]' = \frac{1}{x \ln a}$ for $x > 0$, $a > 0$, $a \neq 1$.
- $[\cos x]' = -\sin x$ for $x \in R$.
- $[\cot x]' = -\frac{1}{\sin^2 x}$ for $x \neq k\pi$, $k \in Z$.
- $[\arccos x]' = -\frac{1}{\sqrt{1-x^2}}$ for $x \in (-1; 1)$.
- $[\operatorname{arccot} x]' = -\frac{1}{1+x^2}$ for $x \in R$.
- $[\cosh x]' = \sinh x$ for $x \in R$.
- $[\coth x]' = -\frac{1}{\sinh^2 x}$ for $x \neq 0$.
- $[\operatorname{arcosh} x]' = \frac{1}{\sqrt{x^2-1}}$ for $x > 1$.
- $[\operatorname{arcoth} x]' = \frac{1}{1-x^2}$ for $x \in R - \langle -1; 1 \rangle$.

For practical needs, it is necessary to remember the formulas from the table.

Derivative of a real function

In the practical calculation of derivatives, we use various formulas and rules.

Rules for derivation.

Functions f, g have derivatives f', g' on the set $A \neq \emptyset$, a point $x_0 \in A$, a number $c \in R$. $\Rightarrow$

- $(cf)'(x_0) = cf'(x_0)$,
- $(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)$,
- $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$,
- $\left[\frac{f}{g}\right]'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g^2(x_0)}$ for $g(x_0) \neq 0$,

- $(cf)' = cf'$.
- $(f \pm g)' = f' \pm g'$.
- $(fg)' = f'g + fg'$.
- $\left[\frac{f}{g}\right]' = \frac{f'g - fg'}{g^2}$.

Functions f, g, h have derivatives f', g', h' on the set $A \neq \emptyset$.

- $[fgh]' = [(fg)h]' = (fg)'h + (fg)h' = [f'g + fg']h + fgh' = f'gh + fg'h + fgh'$.

A function f have derivative $f'(x) \neq 0$ on the set $A \neq \emptyset$.

- $\left[\frac{1}{f(x)}\right]' = \frac{1' \cdot f(x) - 1 \cdot f'(x)}{f^2(x)} = \frac{0 - f'(x)}{f^2(x)} = -\frac{f'(x)}{f^2(x)}$.

Derivative of a real function

In the practical calculation of derivatives, we use various formulas and rules.

Rules for derivation.

Functions f, g have derivatives f', g' on the set $A \neq \emptyset$, a point $x_0 \in A$, a number $c \in R$. $\Rightarrow$

- $(cf)'(x_0) = cf'(x_0)$, • $(cf)' = cf'$.
- $(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)$, • $(f \pm g)' = f' \pm g'$.
- $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$, • $(fg)' = f'g + fg'$.
- $\left[\frac{f}{g}\right]'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g^2(x_0)}$ for $g(x_0) \neq 0$, • $\left[\frac{f}{g}\right]' = \frac{f'g - fg'}{g^2}$.

Functions f, g, h have derivatives f', g', h' on the set $A \neq \emptyset$.

- $[fgh]' = [(fg)h]' = (fg)'h + (fg)h' = [f'g + fg']h + fgh' = f'gh + fg'h + fgh'$.

A function f have derivative $f'(x) \neq 0$ on the set $A \neq \emptyset$.

- $\left[\frac{1}{f(x)}\right]' = \frac{1' \cdot f(x) - 1 \cdot f'(x)}{f^2(x)} = \frac{0 - f'(x)}{f^2(x)} = -\frac{f'(x)}{f^2(x)}$.

Derivative of a real function

The derivative of the inverse function.

The function $y = f(x)$, $x \in I$ is a bijection, $I \subset R$ is an interval, $x_0 \in I$ is an interior point.

$$\left.\begin{array}{l}\bullet\ f \text{ is continuous on } I.\\[1ex] \bullet\ f'(x_0) \neq 0 \text{ is finite.}\end{array}\right\} \Rightarrow \bullet\ \text{Exists } [f^{-1}]'(y_0) = \frac{1}{f'(x_0)}\bigg|_{x_0=f^{-1}(y_0)} = \frac{1}{f'(f^{-1}(y_0))}.$$

The function $f\colon\ y = \mathrm{e}^x$, $x \in R$ is continuous and increasing, $f'(x) = \mathrm{e}^x \neq 0$ for $x \in R$.

- $f^{-1}\colon\ x = \ln y$, $y \in (0; \infty)$.
- $[\ln y]' = [f^{-1}]'(y) = \frac{1}{f'(x)} = \frac{1}{[\mathrm{e}^x]'}\Big|_{x=\ln y} = \frac{1}{\mathrm{e}^x}\Big|_{x=\ln y} = \frac{1}{\mathrm{e}^{\ln y}} = \frac{1}{y}$ for $y \in (0; \infty)$.

The function $f\colon\ y = \sin x$, $x \in \left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$ is continuous and increasing,

$$f'(x) = \cos x = \sqrt{1 - \sin^2 x} > 0 \text{ for } x \in \left(-\tfrac{\pi}{2}; \tfrac{\pi}{2}\right).$$

- $f^{-1}\colon\ x = \arcsin y$, $y \in (-1; 1)$.
- $[\arcsin y]' = \frac{1}{[\sin x]'}\Big|_{x=\arcsin y} = \frac{1}{\cos x}\Big|_{x=\arcsin y} = \frac{1}{\sqrt{1-\sin^2 x}}\Big|_{x=\arcsin y}$ $= \frac{1}{\sqrt{1-[\sin \arcsin y]^2}} = \frac{1}{\sqrt{1-y^2}}$ for $y \in (-1; 1)$.

Derivative of a real function

The derivative of the inverse function.

The function $y=f(x)$, $x\in I$ is a bijection, $I\subset R$ is an interval, $x_0\in I$ is an interior point.

$$\left.\begin{array}{l}\bullet\ f \text{ is continuous on } I.\\ \bullet\ f'(x_0)\neq 0 \text{ is finite.}\end{array}\right\} \Rightarrow \bullet\ \text{Exists } [f^{-1}]'(y_0)=\frac{1}{f'(x_0)}\bigg|_{x_0=f^{-1}(y_0)}=\frac{1}{f'(f^{-1}(y_0))}.$$

The function $f\colon\ y=\mathrm{e}^x,\ x\in R$ is continuous and increasing, $f'(x)=\mathrm{e}^x\neq 0$ for $x\in R$.

- $f^{-1}\colon\ x=\ln y,\ y\in(0;\infty)$.
- $[\ln y]'=[f^{-1}]'(y)=\frac{1}{f'(x)}=\frac{1}{[\mathrm{e}^x]'}\Big|_{x=\ln y}=\frac{1}{\mathrm{e}^x}\Big|_{x=\ln y}=\frac{1}{\mathrm{e}^{\ln y}}=\frac{1}{y}$ for $y\in(0;\infty)$.

The function $f\colon\ y=\sin x,\ x\in\left(-\frac{\pi}{2};\frac{\pi}{2}\right)$ is continuous and increasing,

$$f'(x)=\cos x=\sqrt{1-\sin^2 x}>0 \text{ for } x\in\left(-\tfrac{\pi}{2};\tfrac{\pi}{2}\right).$$

- $f^{-1}\colon\ x=\arcsin y,\ y\in(-1;1)$.
- $[\arcsin y]'=\frac{1}{[\sin x]'}\Big|_{x=\arcsin y}=\frac{1}{\cos x}\Big|_{x=\arcsin y}=\frac{1}{\sqrt{1-\sin^2 x}}\Big|_{x=\arcsin y}$
  $=\frac{1}{\sqrt{1-[\sin\arcsin y]^2}}=\frac{1}{\sqrt{1-y^2}}$ for $y\in(-1;1)$.

beerb@frcatel.fri.uniza.sk http://frcatel.fri.uniza.sk/users/beerb

Derivative of a real function

The derivative of a composite function.

Functions $u=f(x)$, $x\in D(f)$, $y=g(u)$, $u\in D(g)$ so that $H(f)\subset D(g)$,
a composite function $y=F(x)=g(f(x))$, $x\in D(f)$.

$$\left.\begin{array}{l}\bullet\ x_0\in D(f),\ u_0=f(x_0).\\ \bullet\ f'(x_0),\ g'(u_0)\ \text{are finite.}\end{array}\right\}\Rightarrow\ \bullet\ F'(x_0)=[g(f(x_0))]'=g'(f(x_0))\cdot f'(x_0).$$

- $[a^x]'=\left[e^{\ln a^x}\right]'=\left[e^{x\ln a}\right]'=e^{x\ln a}\cdot[x\ln a]'=a^x\cdot\ln a,\ x\in R,\ a>0,\ a\neq 1.$
- $[x^a]'=\left[e^{\ln x^a}\right]'=\left[e^{a\ln x}\right]'=e^{a\ln x}\cdot[a\ln x]'=x^a\cdot\frac{a}{x}=ax^{a-1},\ x>0,\ a\in R.$
- $[x^x]'=\left[e^{\ln x^x}\right]'=\left[e^{x\ln x}\right]'=e^{x\ln x}\cdot[x\ln x]'=x^x\cdot\left[1\cdot\ln x+x\cdot\frac{1}{x}\right]=x^x\cdot[1+\ln x],\ x>0.$
- $[\sin(\sin(\sin x))]'=\cos(\sin(\sin x))\cdot[\sin(\sin x)]'$
  $=\cos(\sin(\sin x))\cdot\cos(\sin x)\cdot[\sin x]'=\cos(\sin(\sin x))\cdot\cos(\sin x)\cdot\cos x,\ x\in R.$

$x\in D(f)$, $f(x)>0$, there exists $f'(x)$. $\Rightarrow\ \left[\ln f(x)\right]'=\frac{f'(x)}{f(x)}$.

$\Rightarrow\ \bullet\ f'(x)=f(x)\cdot\left[\ln f(x)\right]'$ (Logarithmic derivative of the function f).

Derivative of a real function

The derivative of a composite function.

Functions $u = f(x)$, $x \in D(f)$, $y = g(u)$, $u \in D(g)$ so that $H(f) \subset D(g)$,
a composite function $y = F(x) = g(f(x))$, $x \in D(f)$.

$$\left.\begin{array}{l} \bullet\ x_0 \in D(f),\ u_0 = f(x_0). \\ \bullet\ f'(x_0),\ g'(u_0) \text{ are finite.} \end{array}\right\} \Rightarrow \ \bullet\ F'(x_0) = [g(f(x_0))]' = g'(f(x_0)) \cdot f'(x_0).$$

- $[a^x]' = \left[e^{\ln a^x}\right]' = \left[e^{x \ln a}\right]' = e^{x \ln a} \cdot [x \ln a]' = a^x \cdot \ln a$, $x \in R$, $a > 0$, $a \neq 1$.
- $[x^a]' = \left[e^{\ln x^a}\right]' = \left[e^{a \ln x}\right]' = e^{a \ln x} \cdot [a \ln x]' = x^a \cdot \frac{a}{x} = ax^{a-1}$, $x > 0$, $a \in R$.
- $[x^x]' = \left[e^{\ln x^x}\right]' = \left[e^{x \ln x}\right]' = e^{x \ln x} \cdot [x \ln x]' = x^x \cdot \left[1 \cdot \ln x + x \cdot \frac{1}{x}\right] = x^x \cdot [1 + \ln x]$, $x > 0$.
- $[\sin(\sin(\sin x))]' = \cos(\sin(\sin x)) \cdot [\sin(\sin x)]'$
  $= \cos(\sin(\sin x)) \cdot \cos(\sin x) \cdot [\sin x]' = \cos(\sin(\sin x)) \cdot \cos(\sin x) \cdot \cos x$, $x \in R$.

$x \in D(f)$, $f(x) > 0$, there exists $f'(x)$. $\Rightarrow \left[\ln f(x)\right]' = \frac{f'(x)}{f(x)}$.

$\Rightarrow$ • $f'(x) = f(x) \cdot \left[\ln f(x)\right]'$ (Logarithmic derivative of the function f).

Differential of a function and derivatives of higher orders

- We often need to approximate the given function f by another, simpler function g so that the difference $|f(x) - g(x)|$ was as small as possible.
- Mostly, **local approximation** is enough in some neighborhood $O(x_0)$ of the $x_0 \in D(f)$.

A function $y = f(x)$, $x \in D(f)$, a point $x_0 \in D(f)$, there exists a finite $f'(x_0)$.

$$\left.\begin{array}{ll} \bullet\ \mathrm{d}f(x_0, x - x_0) & = f'(x_0) \cdot (x - x_0),\ x \in R. \\ \bullet\ \mathrm{d}f(x_0, h) & = f'(x_0) \cdot h,\ h \in R. \end{array}\right\} \textbf{Differential} \text{ of the function } f \text{ at the point } x_0.$$

Then the function f is called **differentiable** at the point x_0.

The function $f\colon\ y = x$, $x \in R$, $x_0 \in R$, $f'(x_0) = 1$ (finite).

- $\mathrm{d}x = \mathrm{d}f(x_0, h) = f'(x_0) \cdot h = 1 \cdot h = h$, $h \in R$.

A function $y = f(x)$, $x \in D(f)$, a point $x_0 \in D(f)$, where $f'(x_0) \in R$ (finite).

- $\mathrm{d}f(x_0) = \mathrm{d}f(x_0, h) = f'(x_0) \cdot h = f'(x_0) \cdot \mathrm{d}x$, $h \in R$. $\Rightarrow$ • $f'(x_0) = \frac{\mathrm{d}f(x_0)}{\mathrm{d}x}$, resp. $f' = \frac{\mathrm{d}f}{\mathrm{d}x}$.

Differential of a function and derivatives of higher orders

- We often need to approximate the given function f by another, simpler function g so that the difference $|f(x)-g(x)|$ was as small as possible.
- Mostly, **local approximation** is enough in some neighborhood $O(x_0)$ of the $x_0 \in D(f)$.

A function $y=f(x)$, $x\in D(f)$, a point $x_0\in D(f)$, there exists a finite $f'(x_0)$.

$$\left.\begin{aligned} &\bullet\ \mathrm{d}f(x_0, x-x_0) = f'(x_0)\cdot(x-x_0),\ x\in R.\\ &\bullet\ \mathrm{d}f(x_0, h) \qquad\ \ = f'(x_0)\cdot h,\ h\in R.\end{aligned}\right\}\ \textbf{Differential of the function } f \text{ at the point } x_0.$$

Then the function f is called **differentiable** at the point x_0.

The function $f\colon\ y=x$, $x\in R$, $x_0\in R$, $f'(x_0)=1$ (finite).

- $\mathrm{d}x = \mathrm{d}f(x_0, h) = f'(x_0)\cdot h = 1\cdot h = h$, $h\in R$.

A function $y=f(x)$, $x\in D(f)$, a point $x_0\in D(f)$, where $f'(x_0)\in R$ (finite).

- $\mathrm{d}f(x_0) = \mathrm{d}f(x_0,h) = f'(x_0)\cdot h = f'(x_0)\cdot \mathrm{d}x$, $h\in R$. $\Rightarrow$ • $f'(x_0) = \frac{\mathrm{d}f(x_0)}{\mathrm{d}x}$, resp. $f' = \frac{\mathrm{d}f}{\mathrm{d}x}$.

Differential of a function and derivatives of higher orders

The best local linear approximation.

A function $y = f(x)$, $x \in D(f)$ is differentiable at the point $x_0 \in D(f)$.

- Approximation of the function f in the neighborhood $O(x_0)$ using at the point x_0 the tangent line d: $y = f(x_0) + f'(x_0)(x - x_0) = f(x_0)$, $x \in O(x_0)$ is the best of all approximations using a linear function (straight line).

Calculate approximately $\sqrt[6]{1.06}$.

- Let us denote $f(x) = \sqrt[6]{x} = x^{1/6}$, $x > 0$, $x_0 = 1$.
- $f(x_0) = f(1) = 1$.
- $f'(x) = [x^{1/6}]' = \frac{1}{6}x^{-5/6} = \frac{1}{6\sqrt[6]{x^5}}$, $x > 0$.
- $f'(x_0) = f'(1) = \frac{1}{6}$.
- Let $O(1)$ be such that $1.06 \in O(1)$.

$\Rightarrow$ • $\sqrt[6]{x} = f(x) \approx f(1) + f'(1) \cdot (x - 1) = 1 + \frac{x-1}{6} = \frac{6+x-1}{6} = \frac{x+5}{6}$.

$\Rightarrow$ • $\sqrt[6]{1.06} = f(1.06) \approx \frac{1.06+5}{6} = \frac{6.06}{6} = 1.01$.

Exactly $\sqrt[6]{1.06} = 1.0097588$, calculation error < 0.00025.

Differential of a function and derivatives of higher orders

The best local linear approximation.

A function $y = f(x)$, $x \in D(f)$ is differentiable at the point $x_0 \in D(f)$.

- Approximation of the function f in the neighborhood $O(x_0)$ using at the point x_0 the tangent line d: $y = f(x_0) + f'(x_0)(x - x_0) = f(x_0)$, $x \in O(x_0)$ is the best of all approximations using a linear function (straight line).

Calculate approximately $\sqrt[6]{1.06}$.

- Let us denote $f(x) = \sqrt[6]{x} = x^{1/6}$, $x > 0$, $x_0 = 1$.
- $f(x_0) = f(1) = 1$.
- $f'(x) = [x^{1/6}]' = \frac{1}{6}x^{-5/6} = \frac{1}{6\sqrt[6]{x^5}}$, $x > 0$.
- $f'(x_0) = f'(1) = \frac{1}{6}$.
- Let $O(1)$ be such that $1.06 \in O(1)$.

$\Rightarrow$ • $\sqrt[6]{x} = f(x) \approx f(1) + f'(1) \cdot (x - 1) = 1 + \frac{x-1}{6} = \frac{6+x-1}{6} = \frac{x+5}{6}$.

$\Rightarrow$ • $\sqrt[6]{1.06} = f(1.06) \approx \frac{1.06+5}{6} = \frac{6.06}{6} = 1.01$.

Exactly $\sqrt[6]{1.06} = 1.0097588$, calculation error < 0.00025.

Differential of a function and derivatives of higher orders

```
(%i9) c:1.06$ f(x):=x^(1/6)$ s:1$ f1(x):=diff(f(x),x,1)$
      p(x):=f(s)+subst(s,x,f1(x))*(x-s)$
      h(c):=print("c=",c,'f(c),"=",float(f(c)),"approx",
      subst(c,x,float(p(x))))$ fpprintprec:8$ p(x); h(c)$
(%o8) (x-1)/6+1
      c = 1.06  f(1.06) = 1.0097588 approx 1.01
```

The variable `fpprintprec:8` sets the output to 8 digits.
The approximation of the function f makes sense only for x near the point x_0.

```
(%i18) h(.9)$h(1.1)$h(1.2)$h(1.5)$h(2.0)$h(4)$h(9)$h(30)$h(64)$
       c = 0.9  f(0.9) = 0.98259319 approx 0.98333333
       c = 1.1  f(1.1) = 1.0160119 approx 1.0166667
       c = 1.2  f(1.2) = 1.0308533 approx 1.0333333
       c = 1.5  f(1.5) = 1.0699132 approx 1.0833333
       c = 2.0  f(2.0) = 1.122462 approx 1.1666667
       c = 4  f(4) = 1.259921 approx 1.5
       c = 9  f(9) = 1.4422496 approx 2.3333333
       c = 30  f(30) = 1.7627344 approx 5.8333333
       c = 64  f(64) = 2.0 approx 11.5
```

Differential of a function and derivatives of higher orders

```
(%i9)  c:1.06$ f(x):=x^(1/6)$ s:1$ f1(x):=diff(f(x),x,1)$
       p(x):=f(s)+subst(s,x,f1(x))*(x-s)$
       h(c):=print("c=",c,'f(c),"=",float(f(c)),"approx",
       subst(c,x,float(p(x))))$ fpprintprec:8$ p(x); h(c)$
```

(%o8) $\frac{x-1}{6}+1$

$c = 1.06 \quad f(1.06) = 1.0097588$ `approx` 1.01

The variable `fpprintprec:8` sets the output to 8 digits.
The approximation of the function f makes sense only for x near the point x_0.

```
(%i18) h(.9)$h(1.1)$h(1.2)$h(1.5)$h(2.0)$h(4)$h(9)$h(30)$h(64)$
       c = 0.9   f(0.9) = 0.98259319 approx 0.98333333
       c = 1.1   f(1.1) = 1.0160119 approx 1.0166667
       c = 1.2   f(1.2) = 1.0308533 approx 1.0333333
       c = 1.5   f(1.5) = 1.0699132 approx 1.0833333
       c = 2.0   f(2.0) = 1.122462 approx 1.1666667
       c = 4   f(4) = 1.259921 approx 1.5
       c = 9   f(9) = 1.4422496 approx 2.3333333
       c = 30   f(30) = 1.7627344 approx 5.8333333
       c = 64   f(64) = 2.0 approx 11.5
```

Differential of a function and derivatives of higher orders

A function $y = f(x)$, $x \in D(f)$ is differentiable, then (if they exist):

- $y = f'(x) = f^{(1)}(x)$, $x \in A_1 \subset D(f)$, $A_1 \neq \emptyset$.
  First-order derivative (first derivative) of f on the set A_1.
- $y = [f'(x)]' = f''(x) = f^{(2)}(x)$, $x \in A_2 \subset A_1$, $A_2 \neq \emptyset$.
  The second-order derivative (second derivative) of f on the set A_2.
- $y = [f''(x)]' = f'''(x) = f^{(3)}(x)$, $x \in A_3 \subset A_2$, $A_3 \neq \emptyset$.
  Third-order derivative (third derivative) of f on the set A_3. $\ldots$
- $y = [f^{(n-1)}(x)]' = f^{(n)}(x)$, $x \in A_n \subset A_{n-1}$, $A_n \neq \emptyset$.
  The derivative of the n-th order (n-th derivative) of f on the set A_n.

Specially: • $y = f(x) = f^{(0)}(x)$, $x \in D(f)$. Zero derivative (0-th derivative) of f.

The n-th derivative of a function f at the point $x_0 \in D(f)$ (if it exists):

- $f^{(n)}(x_0) = \lim\limits_{x \to x_0} \frac{f^{(n-1)}(x) - f^{(n-1)}(x_0)}{x - x_0} = \lim\limits_{h \to 0} \frac{f^{(n-1)}(x_0+h) - f^{(n-1)}(x_0)}{h}$, $x \in A_n$, $n \in N$.

The function $f^{(n-1)}$ must be defined in some neighborhood of $O(x_0)$.

Differential of a function and derivatives of higher orders

Calculating $f^{(n)}$, $n \in N$ can be very laborious in general.

A function $y = x^k$, $x \in R$, where $k \in N$.

- $[x^k]^{(n)} = k(k-1)\cdots(k-n+1)x^{k-n}$, $x \in R$ for $n = 1, 2, \ldots, k$,
  $$[x^k]' = kx^{k-1},\ [x^k]'' = k(k-1)x^{k-2},\ [x^k]''' = k(k-1)(k-2)x^{k-3},\ \ldots,\ [x^k]^{(k)} = k!.$$
- $[x^k]^{(n)} = 0$, $x \in R$ for $n = k+1, k+2, k+3, \ldots,$
  $$[x^k]^{(k+1)} = [k!]' = 0,\ [x^k]^{(k+2)} = [x^k]^{(k+3)} = [0]' = 0,\ \ldots$$

```
(%i9) f(x,k):=x^k;fn(x,k,n):=diff(f(x,k),x,n)$
      fn(x,k,1);fn(x,k,2);fn(x,k,k);
      fn(x,5,1);fn(x,5,2);fn(x,5,5);fn(x,5,6);
```

(%o1) $f(x,k) := x^k$

(%o3) kx^{k-1}

(%o4) $(k-1)kx^{k-2}$

(%o5) $\frac{\mathrm{d}^k}{\mathrm{d}x^k}x^k$

(%o6) $5x^4$

(%o7) $20x^3$

(%o8) 120

(%o9) 0

beerb@frcatel.fri.uniza.sk http://frcatel.fri.uniza.sk/users/beerb

Differential of a function and derivatives of higher orders

The function $y = \mathrm{e}^x$, $x \in R$. $\Rightarrow$ • $[\mathrm{e}^x]^{(n)} = \mathrm{e}^x$, $x \in R$ for all $n = 0, 1, 2, 3, \ldots$.

The function $y = \sin x$, $x \in R$.

- $[\sin x]' = \cos x,$
  $[\sin x]'' = [\cos x]' = -\sin x,$
  $[\sin x]''' = [\cos x]'' = -\cos x,$
  $[\sin x]^{(4)} = [\cos x]''' = \sin x,$
  $[\sin x]^{(5)} = [\cos x]^{(4)} = \cos x, \ \ldots$
- $[\sin x]^{(n)} = [\sin x]^{(n+4)}$ for $n \in N \cup \{0\}$,
  $[\sin x]^{(2k)} = (-1)^k \sin x,$
  $[\sin x]^{(2k-1)} = (-1)^{k+1} \cos x, \ k \in N.$

The function $y = \cos x$, $x \in R$.

- $[\cos x]' = -\sin x,$
  $[\cos x]'' = [-\sin x]' = -\cos x,$
  $[\cos x]''' = [-\cos x]'' = \sin x,$
  $[\cos x]^{(4)} = [\sin x]''' = \cos x,$
  $[\cos x]^{(5)} = [\cos x]^{(4)} = -\sin x, \ \ldots$
- $[\cos x]^{(n)} = [\cos x]^{(n+4)}$ for $n \in N \cup \{0\}$,
  $[\cos x]^{(2k)} = (-1)^k \cos x,$
  $[\cos x]^{(2k-1)} = (-1)^k \sin x, \ k \in N.$

Leibniz formula.

Functions f, g have derivatives on the set A up to the order $n \in N$ (including).

- $[fg]^{(n)} = \sum\limits_{i=0}^{n} \binom{n}{i} f^{(n-i)} g^{(i)} = \binom{n}{0} f^{(n)} g^{(0)} + \binom{n}{1} f^{(n-1)} g^{(1)} + \cdots + \binom{n}{n} f^{(0)} g^{(n)}.$

Differential of a function and derivatives of higher orders

The function $y = e^x$, $x \in R$. $\Rightarrow$ • $[e^x]^{(n)} = e^x$, $x \in R$ for all $n = 0, 1, 2, 3, \ldots$.

The function $y = \sin x$, $x \in R$.

- $[\sin x]' = \cos x$,
  $[\sin x]'' = [\cos x]' = -\sin x$,
  $[\sin x]''' = [\cos x]'' = -\cos x$,
  $[\sin x]^{(4)} = [\cos x]''' = \sin x$,
  $[\sin x]^{(5)} = [\cos x]^{(4)} = \cos x, \ldots$
- $[\sin x]^{(n)} = [\sin x]^{(n+4)}$ for $n \in N \cup \{0\}$,
  $[\sin x]^{(2k)} = (-1)^k \sin x$,
  $[\sin x]^{(2k-1)} = (-1)^{k+1} \cos x$, $k \in N$.

The function $y = \cos x$, $x \in R$.

- $[\cos x]' = -\sin x$,
  $[\cos x]'' = [-\sin x]' = -\cos x$,
  $[\cos x]''' = [-\cos x]'' = \sin x$,
  $[\cos x]^{(4)} = [\sin x]''' = \cos x$,
  $[\cos x]^{(5)} = [\cos x]^{(4)} = -\sin x, \ldots$
- $[\cos x]^{(n)} = [\cos x]^{(n+4)}$ for $n \in N \cup \{0\}$,
  $[\cos x]^{(2k)} = (-1)^k \cos x$,
  $[\cos x]^{(2k-1)} = (-1)^k \sin x$, $k \in N$.

Leibniz formula.

Functions f, g have derivatives on the set A up to the order $n \in N$ (including).

- $[fg]^{(n)} = \sum_{i=0}^{n} \binom{n}{i} f^{(n-i)} g^{(i)} = \binom{n}{0} f^{(n)} g^{(0)} + \binom{n}{1} f^{(n-1)} g^{(1)} + \cdots + \binom{n}{n} f^{(0)} g^{(n)}$.

beerb@frcatel.fri.uniza.sk http://frcatel.fri.uniza.sk/users/beerb

Applications of the derivative of a function

Theorems about the mean value of a function (Rolle's, Lagrange's) and l'Hospital's rule are among the most common applications of derivation in practice.

Rolle's mean value theorem.

- A function f is continuous on $\langle a; b\rangle$.
- $f(a) = f(b)$.
- $f'(x) \in R^*$ for all $x \in (a; b)$.

$\Rightarrow$

- There exists $c \in (a; b)$: $f'(c) = 0$,
  $c = a + \theta(b - a)$, where $\theta \in (0; 1)$.

Lagrange's mean value theorem.

- A function f is continuous on $\langle a; b\rangle$.
- $f'(x) \in R^*$ for all $x \in (a; b)$.

$\Rightarrow$

- There exists $c \in (a; b)$: $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Let us denote $b = a + h$, $h \in R$, for sufficiently small h we can assume $a + \theta h \approx a$.

- $h = b - a$, $c = a + \theta(b - a) = a + \theta h$, $\theta \in (0; 1)$.
- $f(b) - f(a) = f(a + h) - f(a) = f'(a + \theta h) \cdot h$, $h \in R$, $\theta \in (0; 1)$.
- $f(a + h) = f(a) + f'(a + \theta h) \cdot h \approx f(a) + f'(a)h = f(a) + \mathrm{d}f(a, h)$.

Applications of the derivative of a function

Theorems about the mean value of a function (Rolle's, Lagrange's) and l'Hospital's rule are among the most common applications of derivation in practice.

Rolle's mean value theorem.

$$\left.\begin{array}{l} \bullet \ \text{A function } f \text{ is continuous on } \langle a; b\rangle. \\ \bullet \ f(a) = f(b). \\ \bullet \ f'(x) \in R^* \text{ for all } x \in (a; b). \end{array}\right\} \Rightarrow \begin{array}{r} \bullet \ \text{There exists } c \in (a; b)\text{: } f'(c) = 0, \\ c = a + \theta(b-a), \text{ where } \theta \in (0; 1). \end{array}$$

Lagrange's mean value theorem.

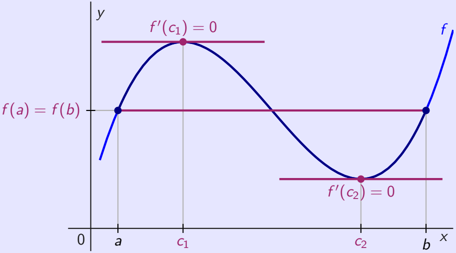
$$\left.\begin{array}{l} \bullet \ \text{A function } f \text{ is continuous on } \langle a; b\rangle. \\ \bullet \ f'(x) \in R^* \text{ for all } x \in (a; b). \end{array}\right\} \Rightarrow \bullet \ \text{There exists } c \in (a; b)\text{: } f'(c) = \frac{f(b)-f(a)}{b-a}.$$

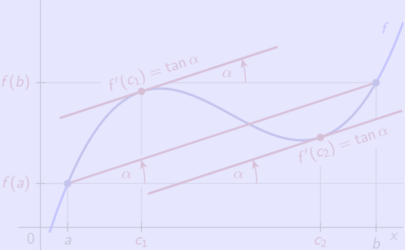
Let us denote $b = a + h$, $h \in R$, for sufficiently small h we can assume $a + \theta h \approx a$.

- $h = b - a$, $c = a + \theta(b - a) = a + \theta h$, $\theta \in (0; 1)$.
- $f(b) - f(a) = f(a + h) - f(a) = f'(a + \theta h) \cdot h$, $h \in R$, $\theta \in (0; 1)$.
- $f(a + h) = f(a) + f'(a + \theta h) \cdot h \approx f(a) + f'(a)h = f(a) + \mathrm{d}f(a, h)$.

Applications of the derivative of a function

Rolle's and Lagrange's mean theorems guarantee the existence of $c \in (a; b)$.
However, we cannot find such points with them, nor can we determine their number.





- $f'(c) = 0$

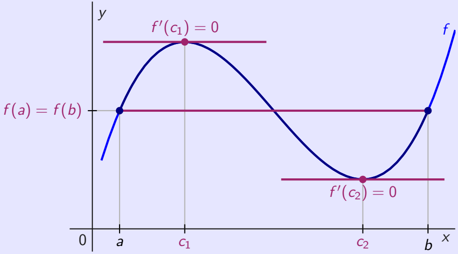
  means that the tangent line to the graph of the function f at the point c is parallel to the x axis.

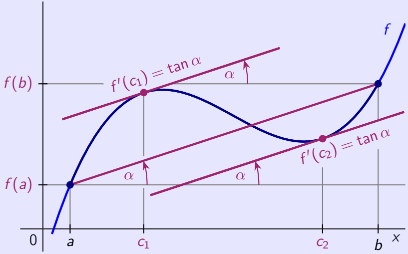
- $f'(c) = \frac{f(b)-f(a)}{b-a}$

  means that the tangent line to the graph of the function f at the point c is parallel to the line connecting the points $[a; f(a)]$ and $[b; f(b)]$, i.e. $f'(c) = \tan\alpha$.

Applications of the derivative of a function

Rolle's and Lagrange's mean theorems guarantee the existence of $c \in (a; b)$.
However, we cannot find such points with them, nor can we determine their number.





- $f'(c) = 0$

  means that the tangent line to the graph of the function f at the point c is parallel to the x axis.

- $f'(c) = \frac{f(b)-f(a)}{b-a}$

  means that the tangent line to the graph of the function f at the point c is parallel to the line connecting the points $[a; f(a)]$ and $[b; f(b)]$, i.e. $f'(c) = \tan\alpha$.

Applications of the derivative of a function

Indefinite expressions of the type $\frac{0}{0}$, resp. $\frac{\infty}{\infty}$ are often they calculate using l'Hospital's rule.

L'Hospital's rule. Functions f, g, a point $a \in R^*$, a neighborhood $O(a)$.

$$\left.\begin{array}{l}
\bullet\ f'(x) \in R^*,\ g'(x) \in R^* \text{ for all } x \in O(a),\ x \neq a.\ . \\
\bullet\ \lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)} = b \in R^*. \\
\left.\begin{array}{l}
\bullet\ \lim\limits_{x \to a} f(x) = \pm\infty,\ \lim\limits_{x \to a} g(x) = \pm\infty\ [L'H\tfrac{\infty}{\infty}]. \\
\text{resp.}\quad \bullet\ \lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x) = 0\ [L'H\tfrac{0}{0}].
\end{array}\right\}
\end{array}\right\} \Rightarrow \bullet\ \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = b.$$

The existence of $\lim\limits_{x \to a} \frac{f(x)}{g(x)}$ does not imply the existence of $\lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$.

- It is very important to verify all the assumptions of l'Hospital's rule.
- Validity of the assumption $\lim\limits_{x \to a} \frac{f'(x)}{g'(x)} = b \in R^*$ is verified continuously during the calculation.
- We can also use L'Hospital's rule several times in a row:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \lim_{x \to a} \frac{f''(x)}{g''(x)} = \cdots = \lim_{x \to a} \frac{f^{(k)}(x)}{g^{(k)}(x)},\ k \in N.$$

Applications of the derivative of a function

A function $y = f(x)$, $x \in D(f)$, a point $x_0 \in D(f)$, $n \in N$,
a neighborhood $O(x_0) \subset D(f)$, $f'(x_0)$, $f''(x_0)$, $\ldots$, $f^{(n)}(x_0) \in R$ (finite).

Taylor polynomial of degree n of the function f centered at the point x_0 is defined as

- $T_n(x) = \sum\limits_{k=0}^{n} \frac{f^{(k)}(x_0)\cdot(x-x_0)^k}{k!} = f(x_0) + \frac{f'(x_0)\cdot(x-x_0)^1}{1!} + \cdots + \frac{f^{(n)}(x_0)\cdot(x-x_0)^n}{n!}$, $x \in O(x_0)$.

For $x_0 = 0$ is called **Maclaurin polynomial**

- $T_n(x) = \sum\limits_{k=0}^{n} \frac{f^{(k)}(0)\cdot x^k}{k!} = f(0) + \frac{f'(0)\cdot x}{1!} + \frac{f''(0)\cdot x^2}{2!} + \cdots + \frac{f^{(n)}(0)\cdot x^n}{n!}$, $x \in O(0)$.

Let us denote $h = x - x_0$, $x = x_0 + h$, $h \in O(0)$.

- $T_n(x_0 + h) = \sum\limits_{k=0}^{n} \frac{f^{(k)}(x_0)\cdot h^k}{k!} = f(x_0) + \frac{f'(x_0)\cdot h}{1!} + \frac{f''(x_0)\cdot h^2}{2!} + \cdots + \frac{f^{(n)}(x_0)\cdot h^n}{n!}$, $h \in O(0)$.

The remainder $R_n(x) = f(x) - T_n(x)$ expresses the approximation error of f using $T_n(x)$.

- $R_n(x) = \frac{f^{(n+1)}(x_0+\theta(x-x_0))\cdot(x-x_0)^{n+1}}{(n+1)!}$, $x \in O(x_0)$, where $\theta \in (0; 1)$. (Lagrangian form.)

Applications of the derivative of a function

A function $y = f(x)$, $x \in D(f)$, a point $x_0 \in D(f)$, $n \in N$,
a neighborhood $O(x_0) \subset D(f)$, $f'(x_0)$, $f''(x_0)$, $\ldots$, $f^{(n)}(x_0) \in R$ (finite).

Taylor polynomial of degree n of the function f centered at the point x_0 is defined as

- $T_n(x) = \sum\limits_{k=0}^{n} \frac{f^{(k)}(x_0)\cdot(x-x_0)^k}{k!} = f(x_0) + \frac{f'(x_0)\cdot(x-x_0)^1}{1!} + \cdots + \frac{f^{(n)}(x_0)\cdot(x-x_0)^n}{n!}$, $x \in O(x_0)$.

For $x_0 = 0$ is called **Maclaurin polynomial**

- $T_n(x) = \sum\limits_{k=0}^{n} \frac{f^{(k)}(0)\cdot x^k}{k!} = f(0) + \frac{f'(0)\cdot x}{1!} + \frac{f''(0)\cdot x^2}{2!} + \cdots + \frac{f^{(n)}(0)\cdot x^n}{n!}$, $x \in O(0)$.

Let us denote $h = x - x_0$, $x = x_0 + h$, $h \in O(0)$.

- $T_n(x_0 + h) = \sum\limits_{k=0}^{n} \frac{f^{(k)}(x_0)\cdot h^k}{k!} = f(x_0) + \frac{f'(x_0)\cdot h}{1!} + \frac{f''(x_0)\cdot h^2}{2!} + \cdots + \frac{f^{(n)}(x_0)\cdot h^n}{n!}$, $h \in O(0)$.

The remainder $R_n(x) = f(x) - T_n(x)$ expresses the approximation error of f using $T_n(x)$.

- $R_n(x) = \frac{f^{(n+1)}(x_0+\theta(x-x_0))\cdot(x-x_0)^{n+1}}{(n+1)!}$, $x \in O(x_0)$, where $\theta \in (0; 1)$. (Lagrangian form.)

Applications of the derivative of a function

The best local approximation using polynomials.

Approximation of f using $T_n(x)$ of degree $n \in N$ in the center $x_0 \in D(f)$:

- It has the local character in a neighborhood $O(x_0)$.
- It is the best of all approximations using degree n polynomials.

$f(x) = \sqrt[3]{1+x} = (x+1)^{\frac{1}{3}}$, $x \in \langle -1; \infty)$, $x_0 = 0$, $f(x_0) = f(0) = 1$.

- $f'(x) = \frac{1}{3}(x+1)^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{(x+1)^2}}$, $x > -1$. • $f'(0) = \frac{1}{3}$.
- $f''(x) = -\frac{2}{3} \cdot \frac{1}{3}(x+1)^{-\frac{5}{3}} = \frac{-2}{9\sqrt[3]{(x+1)^5}}$, $x > -1$. • $f''(0) = -\frac{2}{9}$.
- $f'''(x) = -\frac{5}{3} \cdot (-\frac{2}{9}) \cdot (x+1)^{-\frac{8}{3}} = \frac{10}{27\sqrt[3]{(x+1)^8}}$, $x > -1$. • $f'''(0) = \frac{10}{27}$.

$\Rightarrow$ • $\sqrt[3]{1+x} \approx T_3(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} = 1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81}$, $x \in O(0)$.

$\approx T_2(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} = 1 + \frac{x}{3} - \frac{x^2}{9}$, $x \in O(0)$ with error $R_2(x)$.

$\approx T_1(x) = f(0) + \frac{f'(0)x}{1!} = 1 + \frac{x}{3}$, $x \in O(0)$ with error $R_1(x)$.

Applications of the derivative of a function

The best local approximation using polynomials.

Approximation of f using $T_n(x)$ of degree $n\in N$ in the center $x_0\in D(f)$:

- It has the local character in a neighborhood $O(x_0)$.
- It is the best of all approximations using degree n polynomials.

$f(x)=\sqrt[3]{1+x}=(x+1)^{\frac{1}{3}}$, $x\in\langle -1;\infty)$, $x_0=0$, $f(x_0)=f(0)=1$.

- $f'(x)=\frac{1}{3}(x+1)^{-\frac{2}{3}}=\frac{1}{3\sqrt[3]{(x+1)^2}}$, $x>-1$. • $f'(0)=\frac{1}{3}$.
- $f''(x)=-\frac{2}{3}\cdot\frac{1}{3}(x+1)^{-\frac{5}{3}}=\frac{-2}{9\sqrt[3]{(x+1)^5}}$, $x>-1$. • $f''(0)=-\frac{2}{9}$.
- $f'''(x)=-\frac{5}{3}\cdot(-\frac{2}{9})\cdot(x+1)^{-\frac{8}{3}}=\frac{10}{27\sqrt[3]{(x+1)^8}}$, $x>-1$. • $f'''(0)=\frac{10}{27}$.

$\Rightarrow$ • $\sqrt[3]{1+x}\approx T_3(x)=f(0)+\frac{f'(0)x}{1!}+\frac{f''(0)x^2}{2!}+\frac{f'''(0)x^3}{3!}=1+\frac{x}{3}-\frac{x^2}{9}+\frac{5x^3}{81}$, $x\in O(0)$.

$\approx T_2(x)=f(0)+\frac{f'(0)x}{1!}+\frac{f''(0)x^2}{2!}=1+\frac{x}{3}-\frac{x^2}{9}$, $x\in O(0)$ with error $R_2(x)$.

$\approx T_1(x)=f(0)+\frac{f'(0)x}{1!}=1+\frac{x}{3}$, $x\in O(0)$ with error $R_1(x)$.

Applications of the derivative of a function

We calculate the Taylor polynomial $T_n(x)$ of the function $f(x) = \sqrt{x^2+1}$.

- Manual derivation is quite laborious.

```
(%i2) f(x):=sqrt(x^2+1)$ print("f(x)=", f(x),
      ", f'(x)=", diff(f(x),x),
      ", f''(x)=", ratsimp(diff(f(x),x,2)),
      ", f'''(x)=", ratsimp(diff(f(x),x,3)))$
```

$$f(x) = \sqrt{x^2+1}, \quad f'(x) = \frac{x}{\sqrt{x^2+1}}, \quad f''(x) = \frac{\sqrt{x^2+1}}{x^4+2x^2+1}, \quad f'''(x) = -\frac{3x\sqrt{x^2+1}}{x^6+3x^4+3x^2+1}$$

```
(%i3) taylor(f(x),x,0,1);
```

$$1 + \cdots$$

```
(%i4) taylor(f(x),x,0,2);
```

$$1 + \frac{x^2}{2} + \cdots$$

```
(%i5) taylor(f(x),x,0,3);
```

$$1 + \frac{x^2}{2} + \cdots$$

```
(%i6) taylor(f(x),x,0,4);
```

$$1 + \frac{x^2}{2} - \frac{x^4}{8} + \cdots$$

```
(%i7) taylor(f(x),x,0,18);
```

$$1 + \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{16} - \frac{5x^8}{128} + \frac{7x^{10}}{256} - \frac{21x^{12}}{1024} + \frac{33x^{14}}{2048} - \frac{429x^{16}}{32768} + \frac{715x^{18}}{65536} + \cdots$$

Investigation of behaviour of functions

An important part of the investigation of the behaviour of the function is the determination of the intervals, for which this function is monotonic.

A function f is continuous on an interval I, for all $x \in I$ there exists $f'(x) \in R$ (finite).

The function f is on I	- increasing.	$\Leftrightarrow$	For all $x \in I$ holds	- $f'(x) > 0$.
	- decreasing.	$\Leftrightarrow$		- $f'(x) < 0$.
	- non-decreasing.	$\Leftrightarrow$		- $f'(x) \geq 0$.
	- non-increasing.	$\Leftrightarrow$		- $f'(x) \leq 0$.
	- constant.	$\Leftrightarrow$		- $f'(x) = 0$.

A necessary condition for the existence of a local extremum.

A function $y = f(x)$, $x \in D(f)$, $x_0 \in D(f)$ is an interior point of $D(f)$, there exists $f'(x_0)$.

- The function f has a local extremum at the point x_0. $\Rightarrow$ - $f'(x_0) = 0$.

- The function f can have a local extremum at a point where the derivative does not exist.

- $f'(x_0) = 0$ does not guarantee a local extremum of the function f at the point $x_0 \in D(f)$.

beerb@frcatel.fri.uniza.sk http://frcatel.fri.uniza.sk/users/beerb

Investigation of behaviour of functions

An important part of the investigation of the behaviour of the function is the determination of the intervals, for which this function is monotonic.

A function f is continuous on an interval I, for all $x \in I$ there exists $f'(x) \in R$ (finite).

The function f is on I	• increasing.	$\Leftrightarrow$	For all $x \in I$ holds	• $f'(x) > 0$.
	• decreasing.	$\Leftrightarrow$		• $f'(x) < 0$.
	• non-decreasing.	$\Leftrightarrow$		• $f'(x) \geq 0$.
	• non-increasing.	$\Leftrightarrow$		• $f'(x) \leq 0$.
	• constant.	$\Leftrightarrow$		• $f'(x) = 0$.

A necessary condition for the existence of a local extremum.

A function $y = f(x)$, $x \in D(f)$, $x_0 \in D(f)$ is an interior point of $D(f)$, there exists $f'(x_0)$.

- The function f has a local extremum at the point x_0. $\Rightarrow$ • $f'(x_0) = 0$.

- The function f can have a local extremum at a point where the derivative does not exist.
- $f'(x_0) = 0$ does not guarantee a local extremum of the function f at the point $x_0 \in D(f)$.

Investigation of behaviour of functions

- If $f'(x_0) = 0$ holds, then the point $x_0 \in D(f)$ is called **stationary**.

When searching for local extrema of the function f, we have to investigate:

- All points $x \in D(f)$ for which holds $f'(x) = 0$.
- All points $x \in D(f)$ in which $f'(x)$ does not exist.

When searching for the global extrema of the function f, we must additionally investigate:

- All boundary points $x \in D(f)$.

A sufficient condition for the existence of a local extremum.

A function $y = f(x)$, $x \in D(f)$, $x_0 \in D(f)$, $f'(x_0) = 0$, there exists $f'(x)$ for all $x \in O(x_0)$.

$$\left.\begin{array}{l} \bullet\ f'(x) > 0 \text{ for } x < x_0 \text{ (increasing on the left).} \\ f'(x) < 0 \text{ for } x > x_0 \text{ (decreasing on the right).} \end{array}\right\} \Rightarrow \bullet\ f(x_0) \text{ is a strict local maximum.}$$

$$\left.\begin{array}{l} \bullet\ f'(x) < 0 \text{ for } x < x_0 \text{ (decreasing on the left).} \\ f'(x) > 0 \text{ for } x > x_0 \text{ (increasing on the right).} \end{array}\right\} \Rightarrow \bullet\ f(x_0) \text{ is a strict local minimum.}$$

$$\bullet\ f'(x) > 0, \text{ resp. } f'(x) < 0 \text{ for } x \neq x_0. \qquad \Rightarrow \bullet\ f(x_0) \text{ is not extremum.}$$

Investigation of behaviour of functions

- If $f'(x_0) = 0$ holds, then the point $x_0 \in D(f)$ is called **stationary**.

When searching for local extrema of the function f, we have to investigate:

- All points $x \in D(f)$ for which holds $f'(x) = 0$.
- All points $x \in D(f)$ in which $f'(x)$ does not exist.

When searching for the global extrema of the function f, we must additionally investigate:

- All boundary points $x \in D(f)$.

A sufficient condition for the existence of a local extremum.

A function $y = f(x)$, $x \in D(f)$, $x_0 \in D(f)$, $f'(x_0) = 0$, there exists $f'(x)$ for all $x \in O(x_0)$.

- $\left.\begin{array}{l} f'(x) > 0 \text{ for } x < x_0 \text{ (increasing on the left).} \\ f'(x) < 0 \text{ for } x > x_0 \text{ (decreasing on the right).} \end{array}\right\} \Rightarrow$ • $f(x_0)$ is a strict local maximum.
- $\left.\begin{array}{l} f'(x) < 0 \text{ for } x < x_0 \text{ (decreasing on the left).} \\ f'(x) > 0 \text{ for } x > x_0 \text{ (increasing on the right).} \end{array}\right\} \Rightarrow$ • $f(x_0)$ is a strict local minimum.
- $f'(x) > 0$, resp. $f'(x) < 0$ for $x \neq x_0$. $\Rightarrow$ • $f(x_0)$ is not extremum.

Investigation of behaviour of functions

When investigating local extrema of a function, we can also use its second derivative.

$y = f(x)$, $x \in D(f)$, $x_0 \in D(f)$, $f'(x_0) = 0$, $f''(x_0) \in R - \{0\}$ (finite nonzero).

If $f'(x_0) = 0$ and
- $f''(x_0) < 0. \Rightarrow$ • $f(x_0)$ is a strict local maximum.
- $f''(x_0) > 0. \Rightarrow$ • $f(x_0)$ is a strict local minimum.

$f(x) = x^3 - 6x^2 + 9x - 2$, $x \in R$.

- $f'(x) = 3x^2 - 12x + 9$, $f''(x) = 6x - 12$, $x \in R$. • $f'(x) = 0. \Leftrightarrow x = 1$ or $x = 3$.
  $f''(1) = -6 < 0. \quad \Rightarrow f(1) = 1 - 6 + 9 - 2 = 2 > 0$ is a strict local maximum.
  $f''(3) = 6 > 0. \quad \Rightarrow f(3) = 27 - 54 + 27 - 2 = -2 < 0$ is a strict local minimum.

If $f'(x_0) = f''(x_0) = 0$, then the function f at the point x_0 may or may not have an extremum.

- The function $f(x) = x^3$, $x \in R$ does not have an extremum $f(0) = 0$ at the point $x = 0$.
  $f'(x) = 3x^2$, $f''(x) = 6x$, $f'(0) = f''(0) = 0$.
  $[x^3 < f(0) = 0$ for $x < 0$ and $x^3 > f(0) = 0$ for $x > 0.]$
- The function $f(x) = x^4$, $x \in R$ has a strict local minimum $f(0) = 0$ at the point $x = 0$.
  $f'(x) = 4x^3$, $f''(x) = 12x^2$, $f'(0) = f''(0) = 0$. $[x^4 > f(0) = 0$ for all $x \neq 0.]$

Investigation of behaviour of functions

When investigating local extrema of a function, we can also use its second derivative.

$y = f(x)$, $x \in D(f)$, $x_0 \in D(f)$, $f'(x_0) = 0$, $f''(x_0) \in R - \{0\}$ (finite nonzero).

If $f'(x_0) = 0$ and
- $f''(x_0) < 0. \Rightarrow$ • $f(x_0)$ is a strict local maximum.
- $f''(x_0) > 0. \Rightarrow$ • $f(x_0)$ is a strict local minimum.

$f(x) = x^3 - 6x^2 + 9x - 2,\ x \in R.$

- $f'(x) = 3x^2 - 12x + 9$, $f''(x) = 6x - 12$, $x \in R$. • $f'(x) = 0. \Leftrightarrow x = 1$ or $x = 3$.
  $f''(1) = -6 < 0. \ \Rightarrow f(1) = 1 - 6 + 9 - 2 = 2 > 0$ is a strict local maximum.
  $f''(3) = 6 > 0. \ \Rightarrow f(3) = 27 - 54 + 27 - 2 = -2 < 0$ is a strict local minimum.

If $f'(x_0) = f''(x_0) = 0$, then the function f at the point x_0 may or may not have an extremum.

- The function $f(x) = x^3$, $x \in R$ does not have an extremum $f(0) = 0$ at the point $x = 0$.
  $f'(x) = 3x^2$, $f''(x) = 6x$, $f'(0) = f''(0) = 0$.
  [$x^3 < f(0) = 0$ for $x < 0$ and $x^3 > f(0) = 0$ for $x > 0$.]
- The function $f(x) = x^4$, $x \in R$ has a strict local minimum $f(0) = 0$ at the point $x = 0$.
  $f'(x) = 4x^3$, $f''(x) = 12x^2$, $f'(0) = f''(0) = 0$. [$x^4 > f(0) = 0$ for all $x \neq 0$.]

Investigation of behaviour of functions

A function f is continuous on an interval I, for all $x \in I$ there exists $f'(x) \in R$ (finite).

f is on I	• convex.	$\Leftrightarrow$	f' is on I	• non-decreasing.
	• concave.	$\Leftrightarrow$		• non-increasing.
	• strictly convex.	$\Leftrightarrow$		• increasing.
	• strictly concave.	$\Leftrightarrow$		• descreasing.

A function f is continuous on an interval I, for all $x \in I$ there exists $f''(x) \in R$ (finite).

f is on I	• convex.	$\Leftrightarrow$	For all $x \in I$ holds	• $f''(x) > 0$.
	• concave.	$\Leftrightarrow$		• $f''(x) < 0$.
	• strictly convex.	$\Leftrightarrow$		• $f''(x) \geq 0$.
	• strictly concave.	$\Leftrightarrow$		• $f''(x) \leq 0$.

When investigating the convexity and concavity of the function f, we must examine:

- All points $x \in D(f)$ where the function f is continuous and for which exists $f''(x) = 0$.
- All points $x \in D(f)$ where the function f is continuous and in which $f''(x)$ does not exist.

Investigation of behaviour of functions

A necessary condition for the existence of an inflection point.

A function $y = f(x)$, $x \in D(f)$, $x_0 \in D(f)$ is an interior point of $D(f)$, there exists $f'(x_0)$.

- The point x_0 is the inflection point of the function f. $\Rightarrow$ • $f''(x_0) = 0$.

- The function f can have an inflection at a point where the second derivative does not exist.

A sufficient condition for the existence of a local extremum.

A function $y = f(x)$, $x \in D(f)$, $x_0 \in D(f)$, $f'(x_0) \in R$, there exists $f''(x)$ for all $x \in O(x_0)$.

- $\left.\begin{array}{l} f''(x) > 0 \text{ for } x < x_0 \text{ (convex on the left).} \\ f''(x) < 0 \text{ for } x > x_0 \text{ (concave on the right).} \end{array}\right\} \Rightarrow$ • x_0 is the inflection point of f.
- $\left.\begin{array}{l} f''(x) < 0 \text{ for } x < x_0 \text{ (concave on the left).} \\ f''(x) > 0 \text{ for } x > x_0 \text{ (convex on the right).} \end{array}\right\} \Rightarrow$ • x_0 is the inflection point of f.
- $f''(x) > 0$, resp. $f''(x) < 0$ for $x \neq x_0$. $\Rightarrow$ • x_0 is not an inflection point of f.

$y = f(x)$, $x \in D(f)$, $x_0 \in D(f)$, $f''(x_0) = 0$, $f'''(x_0) \in R$.

- $f'''(x_0) \neq 0$ (non-zero). $\Rightarrow$ • x_0 is the inflection point of f.

Investigation of behaviour of functions

We can generalize the previous results.

$y=f(x)$, $x\in D(f)$, point $x_0\in D(f)$, $n\in N$.

$f'(x_0)=f''(x_0)=\cdots=f^{(n-1)}(x_0)=0$, $f^{(n)}(x_0)\neq 0$.

- $n=2k-1$, $k\in N$ (odd).
  - $f^{(n)}(x_0)>0.\ \Rightarrow$ • f is increasing at the point x_0.
  - $f^{(n)}(x_0)<0.\ \Rightarrow$ • f is decreasing at the point x_0.

  $f(x_0)$ is not extremal.
- $n=2k$, $k\in N$ (even).
  - $f^{(n)}(x_0)>0.\ \Rightarrow$ • $f(x_0)$ is a strict local minimum.
  - $f^{(n)}(x_0)<0.\ \Rightarrow$ • $f(x_0)$ is a strict local maximum.

$y=f(x)$, $x\in D(f)$, point $x_0\in D(f)$, $n\in N$.

$f'(x_0)\in R$, $f''(x_0)=\cdots=f^{(n-1)}(x_0)=0$, $f^{(n)}(x_0)\neq 0$.

- $n=2k+1$, $k\in N$ (odd).
  - x_0 is the inflection point of the function f.
- $n=2k$, $k\in N$ (even).
  - $f^{(n)}(x_0)>0.\ \Rightarrow$ • f is strictly convex at the point x_0.
  - $f^{(n)}(x_0)<0.\ \Rightarrow$ • f is strictly concave at the point x_0.

Behaviour of functions

Investigating the behaviour of the function f means determining:

- Domain $D(f)$, points and intervals of continuity and discontinuity.
- Evenness, oddness, periodicity, respectively other special properties.
- One-sided limits at discontinuity points, boundary points, and $\pm\infty$ points.
- Zero points; intervals on which f is positive and negative.
- f', stationary points, local and global extrema; intervals on which f is increasing, decreasing and constant.
- f'', inflection points; intervals on which f is convex and concave.
- Asymptotes without slope and asymptotes with slope.
- Codomain $H(f)$ and sketch the graph of the function.

  The graph usually gives us the most vivid idea of the behaviour of the function. During its construction, we use all the data found.

  Of course, many times they are insufficient, so we have to supplement them with appropriately chosen functional values.

Indefinite integral

Mathematical Analysis supported by wxMaxima

Basic Terms

- We motivated the introduction of the concept of derivative by the task of determining the instantaneous speed of a mass point, which moves in a straight line.
- We can reverse the problem and look for the trajectory of the mass point provided we know its instantaneous velocity at the given time.

The function $f(x)$, $x \in I$ is defined on the open interval $I \subset R$.
The function $F(x)$, $x \in I$ is called a **primitive function (antiderivative)** to the function $f(x)$ on the interval I, if the derivative $F'(x)$ exists for all $x \in I$ and $F'(x) = f(x)$ holds for all $x \in I$.

A function $F(x)$ is primitive to the function $f(x)$ on the interval I, $c \in R$ (constant).

$\Rightarrow$ • The function $G(x) = F(x) + c$ is primitive to the function $f(x)$ on the interval I.

- It follows from the definition, that primitive function F is continuous on the interval I.

Functions $F(x)$, $G(x)$ are primitive to the function $f(x)$ on the interval I.

$\Rightarrow$ • The function $(F - G)(x) = F(x) - G(x)$ is constant on the interval I.

Basic Terms

- We motivated the introduction of the concept of derivative by the task of determining the instantaneous speed of a mass point, which moves in a straight line.
- We can reverse the problem and look for the trajectory of the mass point provided we know its instantaneous velocity at the given time.

The function $f(x)$, $x \in I$ is defined on the open interval $I \subset R$.
The function $F(x)$, $x \in I$ is called a **primitive function (antiderivative)** to the function $f(x)$ on the interval I, if the derivative $F'(x)$ exists for all $x \in I$ and $F'(x) = f(x)$ holds for all $x \in I$.

A function $F(x)$ is primitive to the function $f(x)$ on the interval I, $c \in R$ (constant).

$\Rightarrow$ • The function $G(x) = F(x) + c$ is primitive to the function $f(x)$ on the interval I.

- It follows from the definition, that primitive function F is continuous on the interval I.

Functions $F(x)$, $G(x)$ are primitive to the function $f(x)$ on the interval I.

$\Rightarrow$ • The function $(F - G)(x) = F(x) - G(x)$ is constant on the interval I.

Basic Terms

All primitive functions to a given function $f(x)$, $x \in I$ on the interval I differ from each other by a constant and form the set $\{F(x) + c, c \in R\}$, where F is any of the primitive functions. This set is called **indefinite integral of the function** f **on the interval** I and is denoted

- $\int f(x)\,\mathrm{d}x = \{F(x) + c, x \in I, c \in R\} = F(x) + c,\ x \in I,\ c \in R.$

$f(x)$, $x \in I$ is continuous on the interval I.

$\Rightarrow$
- There exists $\int f(x)\,\mathrm{d}x$.

The `integrate` command is used to integrate in the wxMaxima.

```
(%i1)  'integrate(1/(1+x^2),x)
```

(%o1) $\int \frac{1}{x^2+1}\,\mathrm{d}x$

Basic Terms

(%i1) `f(x):=1/(1-x^2); integrate(f(x),x);`

(%o1) $\frac{1}{1-x^2}$

(%o2) $\frac{\log(x+1)}{2}-\frac{\log(x-1)}{2}$

- Differentiation and integration are inverse operations on the interval I.

The function F is primitive to the function f on the interval I, $c \in R$.

For all $x \in I$ holds:

- $\left[\int f(x)\,\mathrm{d}x\right]' = \left[F(x)+c\right]' = f(x).$
- $\int F'(x)\,\mathrm{d}x = \int f(x)\,\mathrm{d}x = F(x)+c.$

(%i1) `integrate(1/(1+x^2),x);`

(%o1) $\operatorname{atan} x$

(%i2) `diff(%,x);`

(%o2) $\frac{1}{x^2+1}$

Basic Terms

Indefinite integrals of basic elementary functions. (1st part)

- $\int dx = \int 1\, dx = x + c$ for $x \in R$.
- $\int \frac{dx}{x} = \ln|x| + c$ for $x \neq 0$.
- $\int e^{ax}\, dx = \frac{e^{ax}}{a} + c$ for $a \neq 0,\ x \in R$.
- $\int \sin ax\, dx = -\frac{\cos ax}{a} + c$ for $a \neq 0,\ x \in R$.
- $\int \frac{dx}{\sin^2 ax} = -\frac{\cot ax}{a} + c$ for $a \neq 0,\ x \in R\ x \neq \frac{k\pi}{a},\ k \in Z$.
- $\int \sinh ax\, dx = \frac{\cosh ax}{a} + c$ for $a \neq 0,\ x \in R$.
- $\int \frac{dx}{\sinh^2 ax} = -\frac{\coth ax}{a} + c$ for $a \neq 0,\ x \neq 0$.
- $\int x^a\, dx = \frac{x^{a+1}}{a+1} + c$ for $a \neq -1,\ x \neq 0$.
- $\int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| + c$ for $f(x) \neq 0$.
- $\int a^x\, dx = \frac{a^x}{\ln a} + c$ for $a > 0,\ a \neq 1,\ x \in R$.
- $\int \cos ax\, dx = \frac{\sin ax}{a} + c$ for $a \neq 0,\ x \in R$.
- $\int \frac{dx}{\cos^2 ax} = \frac{\tan ax}{a} + c$ for $a \neq 0,\ x \in R,\ x \neq \frac{(2k+1)\pi}{2a},\ k \in Z$.
- $\int \cosh ax\, dx = \frac{\sinh ax}{a} + c$ for $a \neq 0,\ x \in R$.
- $\int \frac{dx}{\cosh^2 ax} = \frac{\tanh ax}{a} + c$ for $a \neq 0,\ x \in R$.

Basic Terms

Indefinite integrals of basic elementary functions. (2nd part)

- $\int \frac{dx}{x^2+a^2} = \frac{1}{a} \operatorname{arctan} \frac{x}{a} + c_1 = -\frac{1}{a} \operatorname{arccot} \frac{x}{a} + c_2,$ for $a \neq 0,\ x \in R.$
- $\int \frac{dx}{x^2-a^2} = \int \frac{1}{2a} \left[\frac{1}{x-a} - \frac{1}{x+a} \right] dx = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + c,$ for $a \neq 0,\ x \in R - \{a\}.$
- $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin \frac{x}{|a|} + c_1 = -\arccos \frac{x}{|a|} + c_2,$ for $a > 0,\ x \in (-a; a).$
- $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln \left| x + \sqrt{x^2 - a^2} \right| + c,$ for $a > 0,\ x \in (-\infty; -a) \cup (a; \infty).$
- $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln \left(x + \sqrt{x^2 + a^2} \right) + c,$ for $a > 0,\ x \in R.$

- The tables shows the basic formulas for integration.
- These formulas are closely related to formulas for derivatives of elementary functions.
- For practical needs, it is necessary to remember them.

Methods of integration

- $\int \frac{dx}{\sqrt{x^2+1}} = \ln\left(x + \sqrt{x^2+1}\right) + c,\ x \in R.$ (tabular integral).

```
(%i1) integrate(1/sqrt(x^2+1),x);
(%o1) asinh x
```

- Both results are correct because the inverse hyperbolic sine function is defined by $y = \operatorname{arsinh} x = \ln\left(x + \sqrt{x^2+1}\right),\ x \in R$ (see elementary functions).

Decomposition method.

Functions F, G are primitive to functions f, g on the interval I, $a, b \in R$, $|a| + |b| > 0$.

$\Rightarrow$ $aF + bG$ is a primitive to the function $af + bg$ on the interval I and holds:

- $\int \left[af(x) + bg(x)\right] dx = a \int f(x)\, dx + b \int g(x)\, dx = aF(x) + bG(x) + c,\ x \in I,\ c \in R.$

- In practice, we write directly $\int \left[af(x) + bg(x)\right] dx = aF(x) + bG(x) + c.$

Methods of integration

Method per partes.

The functions u, v have continuous derivatives u', v' on the interval I.

$$\Rightarrow \quad \bullet \ \int u(x)\,v'(x)\,\mathrm{d}x = u(x)\,v(x) - \int u'(x)\,v(x)\,\mathrm{d}x,\ x \in I.$$

$$\bullet \ [uv]' = u'v + uv'. \Rightarrow \ \bullet \ uv = \int [uv]' = \int u'v + \int uv'. \Rightarrow \ \bullet \ \int uv' = uv - \int u'v.$$

- We can use the per partes method several times in a row, but we must be careful to stick to it they did not return to the original integral by reuse.
- The per partes method is used quite often. It is suitable for integrating functions

  $$P(x)\,\mathrm{e}^{ax}, \quad P(x)\cos ax, \quad P(x)\sin ax, \quad P(x)\ln Q(x), \quad P(x)\arctan Q(x),$$

  where $P(x)$, $Q(x)$ are real polynomials, $a \in R$, $a \neq 0$.

$$\bullet \ \int \ln x\,\mathrm{d}x = \left[\begin{array}{ll|ll} u & = \ln x & u' & = \frac{1}{x} \\ v' & = 1 & v & = x \end{array}\right] = x\ln x - \int \mathrm{d}x = x\ln x - x + c,\ x \in (0;\infty),\ c \in R.$$

Methods of integration

Substitution method.

A function F is primitive to the function f on the interval I,

$x = \varphi(t)$ has a derivative on the interval J, $\varphi(J) \subset I$.

$\Rightarrow$ $F\big(\varphi(t)\big)$ is primitive to the function $f\big(\varphi(t)\big) \cdot \varphi'(t)$ on J and holds:

- $\displaystyle\int f\big(\varphi(t)\big) \cdot \varphi'(t)\,\mathrm{d}t = \int f(x)\,\mathrm{d}x = F(x) + c = F\big(\varphi(t)\big) + c,\ t \in J,\ c \in R.$

Sets I, J are intervals, $x = \varphi(t) : J \to I$ has a derivative $\varphi'(t) \neq 0$ on J,

a function $F(t)$ is primitive to $f\big(\varphi(t)\big) \cdot \varphi'(t)$ on J.

$\Rightarrow$ $F\big(\varphi^{-1}(x)\big)$ is a primitive to the function $f(x)$ on interval I and holds:

- $\displaystyle\int f(x)\,\mathrm{d}x = \int f\big(\varphi(t)\big) \cdot \varphi'(t)\,\mathrm{d}t = F(t) + c = F\big(\varphi^{-1}(x)\big) + c,\ x \in I,\ c \in R.$

- In the first case we do not have to use inverse substitution,
  but in the second case we have to use the inverse substitution $t = \varphi^{-1}(x)$.

Methods of integration

- $\int \frac{\ln x}{x}\,\mathrm{d}x = \left[\begin{array}{ll} \text{Subst.} & t = \ln x \\ & \mathrm{d}t = \frac{\mathrm{d}x}{x} \end{array}\middle| \begin{array}{l} x \in (0;\infty) \\ t \in R \end{array}\right] = \int t\,\mathrm{d}t = \frac{t^2}{2} + c = \frac{\ln^2 x}{2} + c,\ x \in (0;\infty),\ c \in R.$

- $\int \frac{\ln x}{x}\,\mathrm{d}x = \left[\begin{array}{l} u = \ln x \\ v' = \frac{1}{x} \end{array}\middle| \begin{array}{l} u' = \frac{1}{x} \\ v = \ln x \end{array}\right] = \ln^2 x - \int \frac{\ln x}{x}\,\mathrm{d}x.$

(Equation with the integral as the unknown.)

$\Rightarrow 2\int \frac{\ln x}{x}\,\mathrm{d}x = \ln^2 x + 2c. \Rightarrow$ • $\int \frac{\ln x}{x}\,\mathrm{d}x = \frac{\ln^2 x}{2} + c,\ x > 0,\ c \in R.$

A function $f(x)$ has on the interval I a primitive function $F(x)$, a real number $a, b \in R$, $a \neq 0$.

- $\int f(at+b)\,\mathrm{d}t = \left[\begin{array}{r} \text{Subst. } x = at + b \\ \mathrm{d}x = a\,\mathrm{d}t \end{array}\right] = \int \frac{f(x)\,\mathrm{d}x}{a} = \frac{F(x)}{a} + c = \frac{F(at+b)}{a} + c.$

- $\int f(t+b)\,\mathrm{d}t = \left[\begin{array}{r} \text{Subst. } x = t + b \\ \mathrm{d}x = \mathrm{d}t \end{array}\right] = \int f(x)\,\mathrm{d}x = F(x) + c = F(t+b) + c$ for $a = 1$.

- $\int f(-t)\,\mathrm{d}t = \left[\begin{array}{r} \text{Subst. } x = -t \\ \mathrm{d}x = -\mathrm{d}t \end{array}\right] = -\int f(x)\,\mathrm{d}x = -F(x) + c = -F(-t) + c$ for $a = -1$.

Methods of integration

When integrating, different methods are often combined,
and sometimes they have to be used several times in a row.

If we use different integration procedures, we can arrive at other primitive functions.
(We can verify the correctness of the solution, for example, by derivation.)

- $$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \left[\begin{array}{r|l|l} \text{Subst. } x = \sin t & x \in (-1;1) & \mathrm{d}x = \cos t\,\mathrm{d}t,\ \ (\sin t)' = \cos t > 0 \text{ for } t \in (-\frac{\pi}{2};\frac{\pi}{2}) \\ t = \arcsin x & t \in (-\frac{\pi}{2};\frac{\pi}{2}) & \sqrt{1-x^2} = \sqrt{1-\sin^2 t} = \sqrt{\cos^2 t} = |\cos t| = \cos t \end{array} \right]$$
$$= \int \frac{\cos t\,\mathrm{d}t}{\cos t} = \int \mathrm{d}t = t + c = \arcsin x + c,\ x \in (-1;1),\ c \in R.$$

- $$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \left[\begin{array}{r|l|l} \text{Subst. } x = \cos t & x \in (-1;1) & \mathrm{d}x = -\sin t\,\mathrm{d}t,\ -(\cos t)' = \sin t > 0 \text{ for } t \in (0;\pi) \\ t = \arccos x & t \in (0;\pi) & \sqrt{1-x^2} = \sqrt{1-\cos^2 t} = \sqrt{\sin^2 t} = |\sin t| = \sin t \end{array} \right]$$
$$= \int \frac{-\sin t\,\mathrm{d}t}{\sin t} = -\int \mathrm{d}t = -t + c = -\arccos x + c,\ x \in (-1;1),\ c \in R.$$

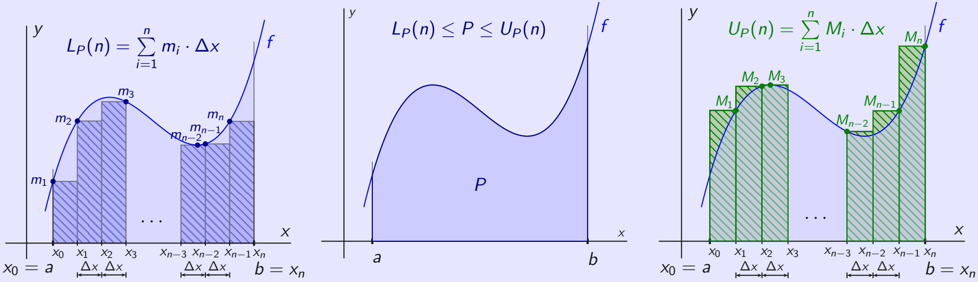
Both solutions are correct because it holds for all $x \in \langle -1;1 \rangle$:
$\arcsin x + \arccos x = \frac{\pi}{2}$, i.e. $\arcsin x = -\arccos x + \frac{\pi}{2}$.
(All primitive functions to a given function on the interval differ by a constant.)

Definite integral

Mathematical Analysis supported by wxMaxima

Basic Terms

- In this section, we will deal with **the definite integral** of the function, which, in contrast to the indefinite integral is not a function, but a specific value (a number or $\pm\infty$).
- We can define a definite integral in several ways.
- We will define it using the so-called integral sums and call **Riemannian** (**definite**) **integral**.



The curvilinear trapezoid P determined by the non-negative function f on the interval $\langle a; b\rangle$ and its approximation using the sums L_P and U_P

Basic Terms

A function $y = f(x)$, $x \in \langle a; b\rangle$ is a positive continuous and points $a, b \in R$, $a < b$.

Determine the area content of the set $P = \left\{[x; y] \in R^2,\ x \in \langle a; b\rangle, 0 \le y \le f(x)\right\}$.

- Let us divide $\langle a; b\rangle$ using points $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$, $n \in N$
  on n subintervals $\langle x_0; x_1\rangle$, $\langle x_1; x_2\rangle$, $\langle x_2; x_3\rangle$, $\ldots$, $\langle x_{n-2}; x_{n-1}\rangle$, $\langle x_{n-1}; x_n\rangle$
  with the same length $\Delta x = x_1 - x_0 = x_2 - x_1 = \cdots = x_n - x_{n-1} = \frac{b-a}{n}$.
- $m_i = \min\{f(x),\ x \in \langle x_{i-1}; x_i\rangle\}$, $M_i = \max\{f(x),\ x \in \langle x_{i-1}; x_i\rangle\}$ for $i = 1, 2, \ldots, n$.

For the area P then holds:

$$\sum_{i=1}^{n} m_i \cdot \Delta x = L_P(n) \le P \le U_P(n) = \sum_{i=1}^{n} M_i \cdot \Delta x.$$

- If we decrease Δx (increase n), the estimates of L_P, U_P will improve (do not get worse).
- For $\Delta x = \frac{b-a}{n} \to 0$, i.e. $n \to \infty$ will hold (Image on previous slide.)

$$\text{(from lower)} \qquad L_P(n) \to P \leftarrow U_P(n) \qquad \text{(from upper)}.$$

Basic Terms

An interval $\langle a; b\rangle$ is a non-degenerate, a function $y = f(x)$, $x \in \langle a; b\rangle$ is a bounded.

- **Dividing the interval** $\langle a; b\rangle$ is called each finite set of points
$$D = \{x_0, x_1, x_2, \ldots, x_n\} = \{x_i\}_{i=0}^{n}, \ \ n \in N,$$
for which $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$.
- Points x_0, x_1, . . . , x_n are called **dividing points** (they uniquely determine the dividing D).
- Interval lengths $d_i = \langle x_{i-1}; x_i\rangle$, $i = 1, 2, \ldots, n$ we denote $\Delta x_i = x_i - x_{i-1}$.
We call the length of the longest of these intervals **dividing norm** D and denote $\mu(D)$, i.e. $\mu(D) = \max\{\Delta x_i,\ i = 1, 2, \ldots, n\}$.
- For the sum of the lengths of the intervals d_1, d_2, . . . , d_n it holds
$$\Delta x_1 + \Delta x_2 + \cdots + \Delta x_n = x_n - x_0 = b - a.$$
- The set of all dividings of the $\langle a; b\rangle$ we denote $\mathfrak{D}_{\langle a;b\rangle} = \{D,\ D \text{ is the division } \langle a; b\rangle\}$.
- $m_i = \inf\{f(x),\ x \in \langle x_{i-1}; x_i\rangle\}, \quad M_i = \sup\{f(x),\ x \in \langle x_{i-1}; x_i\rangle\},\ i = 1, 2, \ldots, n.$
- **Lower** $S_L(f, D)$ and **upper Riemannian integral sum** $S_U(f, D)$ of function f under dividing D are called numbers $S_L(f, D) = \sum\limits_{i=1}^{n} m_i \cdot \Delta x_i$ and $S_U(f, D) = \sum\limits_{i=1}^{n} M_i \cdot \Delta x_i$.

Basic Terms

- Numbers
  $$\underline{\int_a^b} f(x)\,\mathrm{d}x = \sup\left\{S_L(f,D),\ D\in\mathfrak{D}_{\langle a;b\rangle}\right\},\ \overline{\int_a^b} f(x)\,\mathrm{d}x = \inf\left\{S_U(f,D),\ D\in\mathfrak{D}_{\langle a;b\rangle}\right\}$$
  we call the **lower** and **upper Riemannian (definite) integral** of the function f on the interval $\langle a;b\rangle$ (from a to b).
- These numbers **always exist** and holds
  $$m(b-a) \leq \underline{\int_a^b} f(x)\,\mathrm{d}x \leq \overline{\int_a^b} f(x)\,\mathrm{d}x \leq M(b-a),$$
  while $m = \inf\{f(x),\ x\in\langle a;b\rangle\}$, $M = \sup\{f(x),\ x\in\langle a;b\rangle\}$.
- If the equality between the lower and upper Riemann integrals holds, then this value
  $$\int_a^b f(x)\,\mathrm{d}x = \underline{\int_a^b} f(x)\,\mathrm{d}x = \overline{\int_a^b} f(x)\,\mathrm{d}x$$
  we call the **Riemannian (definite) integral of the function** f **on the interval** $\langle a;b\rangle$.
  We call the function f **Riemannian integrable** on the $\langle a;b\rangle$ and we denote $f\in R_{\langle a;b\rangle}$.

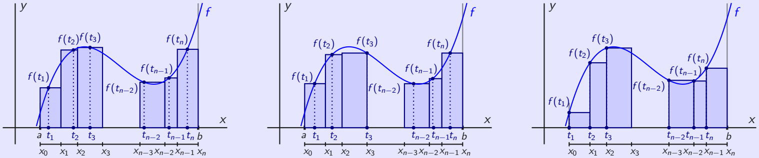
Basic Terms

The integration variable has no effect and instead of x we can write any symbol.

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^b f(t)\,\mathrm{d}t = \int_a^b f(u)\,\mathrm{d}u = \int_a^b f(z)\,\mathrm{d}z = \int_a^b f(\varphi)\,\mathrm{d}\varphi.$$

- **Riemannian (integral) sum** function f under dividing D and choice of points T, where $T = \{t_1, t_2, \ldots, t_n\} = \{t_i,\ t_i \in \langle x_{i-1}; x_i\rangle\}_{i=1}^{n}$ we call the number
$$S_T(f, D) = \sum_{i=1}^{n} f(t_i) \cdot \Delta x_i.$$
- The function f has infinitely many integral sums for a given dividing D.

  If $y = f(x)$, $x \in \langle a; b\rangle$ is continuous, then f takes its extrema on each interval $\langle x_{i-1}; x_i\rangle$, $i = 1, 2, \ldots, n$ and $S_L(f, D)$ and $S_U(f, D)$ are also Riemann integral sums for some particular choices of points T.



Integral sums of the function f under dividing D and different choices of points T

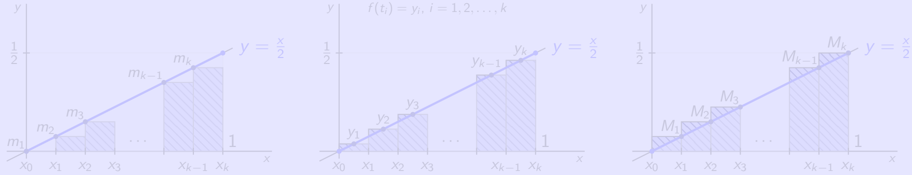
Basic Terms

- When investigating the Riemann integrable function f on the interval $\langle a; b\rangle$, we do not need all the dividing $D \in \mathfrak{D}_{\langle a;b\rangle}$.

  It is sufficient to restrict ourselves to **normal sequences** of dividing $\{D_k\}_{k=1}^{\infty} \subset \mathfrak{D}_{\langle a;b\rangle}$, i.e. if $\lim\limits_{k\to\infty} \mu(D_k) = 0$ holds.

  Then it holds for every choice of points T

$$\lim_{k\to\infty} S_T(f, D_k) = \int_a^b f(x)\,\mathrm{d}x.$$



$S_L(f, D_k) = \sum\limits_{i=1}^{k} m_i \cdot \Delta x_i = \sum\limits_{i=1}^{k} \frac{i-1}{2k\cdot k} = \frac{k-1}{4k}$ $S_T(f, D_k) = \sum\limits_{i=1}^{k} f(t_i) \cdot \Delta x_i = \sum\limits_{i=1}^{k} \frac{2i-1}{4k\cdot k} = \frac{1}{4}$ $S_U(f, D_k) = \sum\limits_{i=1}^{k} M_i \cdot \Delta x_i = \sum\limits_{i=1}^{k} \frac{i}{2k\cdot k} = \frac{k+1}{4k}$

$\int_0^1 \frac{x\,\mathrm{d}x}{2} = \frac{1}{4}$ (next page).

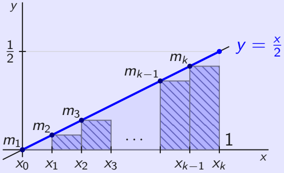
Basic Terms

- When investigating the Riemann integrable function f on the interval $\langle a; b\rangle$, we do not need all the dividing $D \in \mathfrak{D}_{\langle a;b\rangle}$.

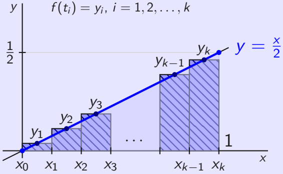
  It is sufficient to restrict ourselves to **normal sequences** of dividing $\{D_k\}_{k=1}^{\infty} \subset \mathfrak{D}_{\langle a;b\rangle}$, i.e. if $\lim\limits_{k\to\infty} \mu(D_k) = 0$ holds.

  Then it holds for every choice of points T

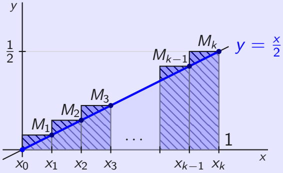
$$\lim_{k\to\infty} S_T(f, D_k) = \int_a^b f(x)\,\mathrm{d}x.$$



$S_L(f, D_k) = \sum_{i=1}^{k} m_i \cdot \Delta x_i = \sum_{i=1}^{k} \frac{i-1}{2k\cdot k} = \frac{k-1}{4k}$



$S_T(f, D_k) = \sum_{i=1}^{k} f(t_i) \cdot \Delta x_i = \sum_{i=1}^{k} \frac{2i-1}{4k\cdot k} = \frac{1}{4}$



$S_U(f, D_k) = \sum_{i=1}^{k} M_i \cdot \Delta x_i = \sum_{i=1}^{k} \frac{i}{2k\cdot k} = \frac{k+1}{4k}$

$\int_0^1 \frac{x\,\mathrm{d}x}{2} = \frac{1}{4}$ (next page).

Basic Terms

$\int_0^1 \frac{x\,\mathrm{d}x}{2} = \frac{1}{4}$. The function $f(x) = \frac{x}{2}$, $x \in \langle 0; 1\rangle$ is increasing, continuous, $f \in R_{\langle 0;1\rangle}$.

- A normal sequence of dividings $\{D_k\}_{k=1}^{\infty} \subset \mathfrak{D}_{\langle 0;1\rangle}$, while $D_k = \left\{\frac{i}{k}\right\}_{i=0}^{k}$ for $k \in N$.
- For $i = 1, 2, \ldots, k$ holds $\Delta x_i = \frac{1}{k}$, $m_i = f(x_{i-1}) = \frac{i-1}{2k}$, $M_i = f(x_i) = \frac{i}{2k}$.

$$S_L(f, D_k) = \sum_{i=1}^{k} m_i \cdot \Delta x_i = \sum_{i=1}^{k} \frac{i-1}{2k} \cdot \frac{1}{k} = \frac{0+1+\cdots+(k-1)}{2k^2} = \frac{\frac{(0+k-1)k}{2}}{2k^2} = \frac{k-1}{4k} = \frac{1}{4} - \frac{1}{4k}.$$

$$S_U(f, D_k) = \sum_{i=1}^{k} M_i \cdot \Delta x_i = \sum_{i=1}^{k} \frac{i}{2k} \cdot \frac{1}{k} = \frac{1+2+\cdots+k}{2k^2} = \frac{\frac{(1+k)k}{2}}{2k^2} = \frac{k+1}{4k} = \frac{1}{4} + \frac{1}{4k}.$$

$$\Rightarrow \bullet \int_0^1 \frac{x\,\mathrm{d}x}{2} = \lim_{k\to\infty} S_L(f, D_k) = \lim_{k\to\infty} S_U(f, D_k) = \lim_{k\to\infty} \left(\frac{1}{4} \pm \frac{1}{4k}\right) = \frac{1}{4}.$$

- Let's choose $T = \{t_i\}_{i=1}^{k}$ as points centers of intervals $\langle x_{i-1}; x_i\rangle$, $i = 1, 2, \ldots, k$,
  i.e. $t_i = \frac{1}{2}\left(\frac{i-1}{k} + \frac{i}{2}\right) = \frac{2i-1}{2k}$, then $f(t_i) = \frac{2i-1}{4k}$ and holds

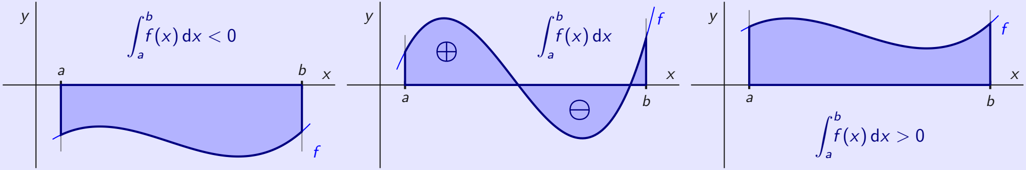
$$S_T(f, D_k) = \sum_{i=1}^{k} f(t_i) \cdot \Delta x_i = \sum_{i=1}^{k} \frac{2i-1}{4k} \cdot \frac{1}{k} = \frac{1+3+\cdots+(2k-1)}{4k^2} = \frac{\frac{(1+2k-1)k}{2}}{4k^2} = \frac{1}{4}.$$

$$\Rightarrow \bullet \int_0^1 \frac{x\,\mathrm{d}x}{2} = \lim_{k\to\infty} S_T(f, D_k) = \lim_{k\to\infty} \frac{1}{4} = \frac{1}{4}.$$

Basic Properties

- Geometrically, it represents the Riemannian definite integral on the interval $\langle a; b\rangle$ the area of the curvilinear trapezoid determined by the function f and the interval $\langle a; b\rangle$.

  Below the x axis (i.e., if f is negative), this area is negative.



Functions $f, g \in R_{\langle a;b\rangle}$, a number $c \in R$.

$\Rightarrow cf, f+g, |f|, f^2, fg \in R_{\langle a;b\rangle}$ and holds:

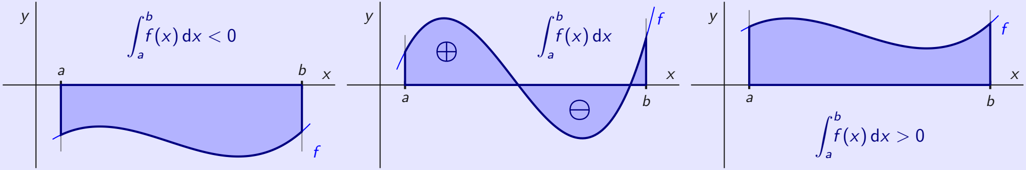
- $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx,$
- $\displaystyle\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$

If $\inf\{g(x), x \in \langle a; b\rangle\} > 0$, resp. $\sup\{g(x), x \in \langle a; b\rangle\} < 0$, then also $\frac{1}{g}, \frac{f}{g} \in R_{\langle a;b\rangle}$.

Basic Properties

- Geometrically, it represents the Riemannian definite integral on the interval $\langle a; b\rangle$ the area of the curvilinear trapezoid determined by the function f and the interval $\langle a; b\rangle$.

  Below the x axis (i.e., if f is negative), this area is negative.



Functions $f, g \in R_{\langle a;b\rangle}$, a number $c \in R$.

$\Rightarrow cf, f+g, |f|, f^2, fg \in R_{\langle a;b\rangle}$ and holds:

- $\displaystyle\int_a^b cf(x)\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x,$
- $\displaystyle\int_a^b [f(x)+g(x)]\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x.$

If $\inf\{g(x), x \in \langle a; b\rangle\} > 0$, resp. $\sup\{g(x), x \in \langle a; b\rangle\} < 0$, then also $\frac{1}{g}, \frac{f}{g} \in R_{\langle a;b\rangle}$.

Basic Properties

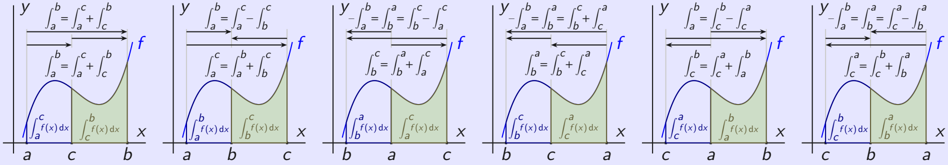
Functions $f, g \in R_{\langle a;b\rangle}$.

- $f(x) \geq 0$ for all $x \in \langle a; b\rangle$. $\Rightarrow$ • $\displaystyle\int_a^b f(x)\,\mathrm{d}x \geq 0.$
- $g(x) \geq f(x)$ for all $x \in \langle a; b\rangle$. $\Rightarrow$ • $\displaystyle\int_a^b g(x)\,\mathrm{d}x \geq \int_a^b f(x)\,\mathrm{d}x.$

Additivity of the integral.

A function $f \in R_I$, $I \subset R$ is a bounded interval, points $a, b, c \in I$ are arbitrary.

$\Rightarrow$ • $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x.$



$a < c < b$ $a < b < c$ $b < a < c$ $b < c < a$ $c < a < b$ $c < b < a$

We can clearly illustrate the additivity of the Riemann integral on vectors.

Methods of integration

Calculation of the Riemann integral (Newton-Leibniz formula).

A function $f \in R_{\langle a;b\rangle}$, the function F is a primitive function to the function f on $\langle a; b\rangle$.

$\Rightarrow$

- $$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a) = \Big[F(x)\Big]_a^b.$$

- $$\int_{-1}^{0} \tfrac{x}{2}\,\mathrm{d}x = \left[\tfrac{x^2}{2\cdot 2}\right]_{-1}^{0} = \left[\tfrac{x^2}{4}\right]_{-1}^{0} = \tfrac{0^2}{4} - \tfrac{(-1)^2}{4} = \tfrac{1}{4}.$$

- $$\int_{-1}^{1} x^2\,\mathrm{d}x = \left[\tfrac{x^3}{3}\right]_{-1}^{1} = \tfrac{1^2}{3} - \tfrac{(-1)^3}{3} = \tfrac{2}{3}.$$

- $$\int_{0}^{1} \tfrac{\mathrm{d}x}{\sqrt{x^2+1}} = \left[\ln\left(x+\sqrt{x^2+1}\right)\right]_0^1 = \ln\left(1+\sqrt{2}\right) - \ln 1 = \ln\left(1+\sqrt{2}\right).$$

```
(%i1) integrate(f(x),x,-1,1);
```

(%o1) $\int_{-1}^{1} f(x)\,\mathrm{d}x$

Methods of integration

Calculation of the Riemann integral (Newton-Leibniz formula).

A function $f \in R_{\langle a;b\rangle}$, the function F is a primitive function to the function f on $\langle a; b\rangle$.

$\Rightarrow$
- $\displaystyle\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a) = \Big[F(x)\Big]_a^b.$

- $$\int_{-1}^{0} \tfrac{x}{2}\,\mathrm{d}x = \left[\tfrac{x^2}{2\cdot 2}\right]_{-1}^{0} = \left[\tfrac{x^2}{4}\right]_{-1}^{0} = \tfrac{0^2}{4} - \tfrac{(-1)^2}{4} = \tfrac{1}{4}.$$
- $$\int_{-1}^{1} x^2\,\mathrm{d}x = \left[\tfrac{x^3}{3}\right]_{-1}^{1} = \tfrac{1^2}{3} - \tfrac{(-1)^3}{3} = \tfrac{2}{3}.$$
- $$\int_{0}^{1} \tfrac{\mathrm{d}x}{\sqrt{x^2+1}} = \left[\ln\left(x+\sqrt{x^2+1}\right)\right]_0^1 = \ln\left(1+\sqrt{2}\right) - \ln 1 = \ln\left(1+\sqrt{2}\right).$$

```
(%i1) integrate(f(x),x,-1,1);
```

(%o1) $\displaystyle\int_{-1}^{1} f(x)\,\mathrm{d}x$

Methods of integration

- Definite integrals are generally calculated using indefinite integrals.
- We can modify the per partes method and substitution methods and calculate the definite integral using them directly.

  After substitution, we do not need to return to the original variables.

Method per partes.

$$u, u', v, v' \in R_{\langle a;b\rangle}. \Rightarrow \quad \bullet \int_a^b u(x)\, v'(x)\, \mathrm{d}x = \Big[u(x)\, v(x)\Big]_a^b - \int_a^b u'(x)\, v(x)\, \mathrm{d}x.$$

$$\int_0^{2\pi} x^2 \sin x \,\mathrm{d}x = \left[\begin{array}{l|l} u = x^2 & u' = 2x \\ v' = \sin x & v = -\cos x \end{array}\right] = \Big[-x^2 \cos x\Big]_0^{2\pi} + \int_0^{2\pi} 2x \cos x \,\mathrm{d}x$$

$$= \left[\begin{array}{l|l} u = 2x & u' = 2 \\ v' = \cos x & v = \sin x \end{array}\right] = \Big[-4\pi^2 \cdot 1 + 0^2 \cdot 1\Big] + \Big[2x \sin x\Big]_0^{2\pi} - \int_0^{2\pi} 2 \sin x \,\mathrm{d}x$$

$$= -4\pi^2 + \Big[4\pi \cdot 0 - 2 \cdot 0 \cdot 0\Big] - \Big[-2\cos x\Big]_0^{2\pi} = -4\pi^2 - \Big[-2 \cdot 1 + 2 \cdot 1\Big] = -4\pi^2.$$

Methods of integration

Substitution method. $y=f(x)\colon I\to R,\ x=\varphi(t)\colon J\to R.$

f is continuous on I, φ' is continuous on J, $\varphi(J)\subset I$,
I is an interval with boundaries a, b, J is an interval with boundaries α,β, $\varphi(\alpha)=a$, $\varphi(\beta)=b$.
$\Rightarrow f(\varphi)\varphi'\in R_J$ and holds • $\displaystyle\int_a^b f(x)\,\mathrm{d}x=\int_\alpha^\beta f[\varphi(t)]\varphi'(t)\,\mathrm{d}t.$ (We can use in both directions.)

- $$\int_{-1}^{1}\sqrt{1-x^2}\,\mathrm{d}x\left[\begin{array}{r|l|c|c}\text{Subst. } x=\sin t & x\in\langle -1;1\rangle & 1=\sin\frac{\pi}{2} & \sqrt{1-x^2}=\sqrt{1-\sin^2 t}=\sqrt{\cos^2 t}=|\cos t|=\cos t\\ \mathrm{d}x=\cos t\,\mathrm{d}t & t\in\left\langle -\frac{\pi}{2};\frac{\pi}{2}\right\rangle & -1=\sin\left(-\frac{\pi}{2}\right) & \cos t\geq 0 \text{ pre všetky } t\in\left\langle -\frac{\pi}{2};\frac{\pi}{2}\right\rangle\end{array}\right]$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2 t\,\mathrm{d}t=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\tfrac{1+\cos 2t}{2}\,\mathrm{d}t=\tfrac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\big[1+\cos 2t\big]\,\mathrm{d}t=\tfrac{1}{2}\Big[t+\tfrac{\sin 2t}{2}\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$=\tfrac{1}{2}\Big[\tfrac{\pi}{2}+\tfrac{\sin\pi}{2}-\Big(-\tfrac{\pi}{2}+\tfrac{\sin(-\pi)}{2}\Big)\Big]=\tfrac{1}{2}\Big[\tfrac{\pi}{2}+0-\big(-\tfrac{\pi}{2}+0\big)\Big]=\tfrac{1}{2}\cdot\pi=\tfrac{\pi}{2}.$$

- $$\int_{-1}^{2}t\sin\left(t^2+1\right)\mathrm{d}t=\left[\begin{array}{r|l|l|l}\text{Subst. } x=t^2+1 & t\in\langle -1;0\rangle & x\in\langle 1;2\rangle & t=2\ \ \mapsto x=5\\ \mathrm{d}x=2\,\mathrm{d}t & t\in\langle 0;2\rangle & x\in\langle 1;5\rangle & t=-1\mapsto x=2\end{array}\right]=\tfrac{1}{2}\int_2^5\sin x\,\mathrm{d}x$$

$$=\tfrac{1}{2}\Big[-\cos x\Big]_2^5=\tfrac{1}{2}\Big[-\cos 5+\cos 2\Big]=\tfrac{\cos 2-\cos 5}{2}.$$

Methods of integration

Substitution method. $y = f(x)\colon I \to R,\ x = \varphi(t)\colon J \to R.$

f is continuous on I, φ' is continuous on J, $\varphi(J) \subset I$,
I is an interval with boundaries a, b, J is an interval with boundaries α, β, $\varphi(\alpha) = a$, $\varphi(\beta) = b$.
$\Rightarrow f(\varphi)\varphi' \in R_J$ and holds • $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \int_\alpha^\beta f[\varphi(t)]\varphi'(t)\,\mathrm{d}t.$ (We can use in both directions.)

- $$\int_{-1}^{1}\sqrt{1-x^2}\,\mathrm{d}x\left[\begin{array}{r|l|r|c}\text{Subst. } x = \sin t & x \in \langle -1;1\rangle & 1 = \sin\frac{\pi}{2} & \sqrt{1-x^2} = \sqrt{1-\sin^2 t} = \sqrt{\cos^2 t} = |\cos t| = \cos t \\ \mathrm{d}x = \cos t\,\mathrm{d}t & t \in \left\langle -\frac{\pi}{2};\frac{\pi}{2}\right\rangle & -1 = \sin\left(-\frac{\pi}{2}\right) & \cos t \geq 0 \text{ pre všetky } t \in \left\langle -\frac{\pi}{2};\frac{\pi}{2}\right\rangle\end{array}\right]$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2 t\,\mathrm{d}t = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1+\cos 2t}{2}\,\mathrm{d}t = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[1+\cos 2t\right]\mathrm{d}t = \frac{1}{2}\left[t + \frac{\sin 2t}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\frac{\pi}{2} + \frac{\sin\pi}{2} - \left(-\frac{\pi}{2} + \frac{\sin(-\pi)}{2}\right)\right] = \frac{1}{2}\left[\frac{\pi}{2} + 0 - \left(-\frac{\pi}{2} + 0\right)\right] = \frac{1}{2}\cdot\pi = \frac{\pi}{2}.$$

- $$\int_{-1}^{2} t\sin(t^2+1)\,\mathrm{d}t = \left[\begin{array}{r|l|l|l}\text{Subst. } x = t^2+1 & t \in \langle -1;0\rangle & x \in \langle 1;2\rangle & t = 2 \mapsto x = 5 \\ \mathrm{d}x = 2\,\mathrm{d}t & t \in \langle 0;2\rangle & x \in \langle 1;5\rangle & t = -1 \mapsto x = 2\end{array}\right] = \frac{1}{2}\int_2^5 \sin x\,\mathrm{d}x$$

$$= \frac{1}{2}\Big[-\cos x\Big]_2^5 = \frac{1}{2}\Big[-\cos 5 + \cos 2\Big] = \frac{\cos 2 - \cos 5}{2}.$$

Methods of integration

Substitution method. $y=f(x)\colon I\to R,\ x=\varphi(t)\colon J\to R.$

f is continuous on I, φ' is continuous on J, $\varphi(J)\subset I$,
I is an interval with boundaries a, b, J is an interval with boundaries α,β, $\varphi(\alpha)=a$, $\varphi(\beta)=b$.
$\Rightarrow f(\varphi)\varphi'\in R_J$ and holds • $\displaystyle\int_a^b f(x)\,dx=\int_\alpha^\beta f[\varphi(t)]\varphi'(t)\,dt.$ (We can use in both directions.)

- $$\int_{-1}^{1}\sqrt{1-x^2}\,dx\left[\begin{array}{r|l|c|c} \text{Subst. } x=\sin t & x\in\langle -1;1\rangle & 1=\sin\frac{\pi}{2} & \sqrt{1-x^2}=\sqrt{1-\sin^2 t}=\sqrt{\cos^2 t}=|\cos t|=\cos t \\ dx=\cos t\,dt & t\in\left\langle -\frac{\pi}{2};\frac{\pi}{2}\right\rangle & -1=\sin\left(-\frac{\pi}{2}\right) & \cos t\geq 0 \text{ pre všetky } t\in\left\langle -\frac{\pi}{2};\frac{\pi}{2}\right\rangle \end{array}\right]$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2 t\,dt=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1+\cos 2t}{2}\,dt=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[1+\cos 2t\right]dt=\frac{1}{2}\left[t+\frac{\sin 2t}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$=\frac{1}{2}\left[\frac{\pi}{2}+\frac{\sin\pi}{2}-\left(-\frac{\pi}{2}+\frac{\sin(-\pi)}{2}\right)\right]=\frac{1}{2}\left[\frac{\pi}{2}+0-\left(-\frac{\pi}{2}+0\right)\right]=\frac{1}{2}\cdot\pi=\frac{\pi}{2}.$$

- $$\int_{-1}^{2}t\sin\left(t^2+1\right)dt=\left[\begin{array}{r|l|l|l} \text{Subst. } x=t^2+1 & t\in\langle -1;0\rangle & x\in\langle 1;2\rangle & t=2\ \ \mapsto x=5 \\ dx=2\,dt & t\in\langle 0;2\rangle & x\in\langle 1;5\rangle & t=-1\mapsto x=2 \end{array}\right]=\frac{1}{2}\int_{2}^{5}\sin x\,dx$$

$$=\frac{1}{2}\Big[-\cos x\Big]_2^5=\frac{1}{2}\Big[-\cos 5+\cos 2\Big]=\frac{\cos 2-\cos 5}{2}.$$

Integrating even and odd functions

A function $f \in R_{\langle a;b\rangle}$ is even or odd, where $a < b$.

$\Rightarrow f(-x) \in R_{\langle -b;-a\rangle}$ and holds:

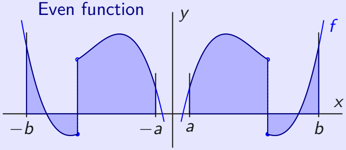
- $$\int_a^b f(x)\,\mathrm{d}x = \left[\begin{matrix} \text{Subst. } t=-x \\ \mathrm{d}t = -\,\mathrm{d}x \end{matrix} \middle| \begin{matrix} x=b \mapsto t=-b \\ x=a \mapsto t=-a \end{matrix}\right] = -\int_{-a}^{-b} f(-t)\,\mathrm{d}t = \int_{-b}^{-a} f(-t)\,\mathrm{d}t = \int_{-b}^{-a} f(-x)\,\mathrm{d}x.$$

---

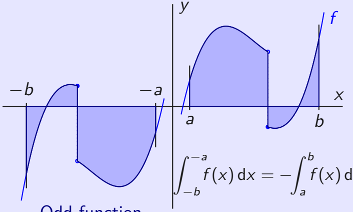
f is even. $\Rightarrow$ • $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \int_{-b}^{-a} f(-x)\,\mathrm{d}x = \int_{-b}^{-a} f(x)\,\mathrm{d}x.$

f is odd. $\Rightarrow$ • $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \int_{-b}^{-a} f(-x)\,\mathrm{d}x = \int_{-b}^{-a} \big[-f(x)\big]\,\mathrm{d}x = -\int_{-b}^{-a} f(x)\,\mathrm{d}x.$

Even function



$$\int_{-b}^{-a} f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x$$



$$\int_{-b}^{-a} f(x)\,\mathrm{d}x = -\int_a^b f(x)\,\mathrm{d}x$$

Odd function

Integrating even and odd functions

A function $f \in R_{\langle -a;a\rangle}$, where $a > 0$.

f is odd. $\Rightarrow$ • $\displaystyle\int_{-a}^{a} f(x)\,\mathrm{d}x = \int_{-a}^{0} f(x)\,\mathrm{d}x + \int_{0}^{a} f(x)\,\mathrm{d}x = -\int_{0}^{-(-a)} f(x)\,\mathrm{d}x + \int_{0}^{a} f(x)\,\mathrm{d}x = 0.$

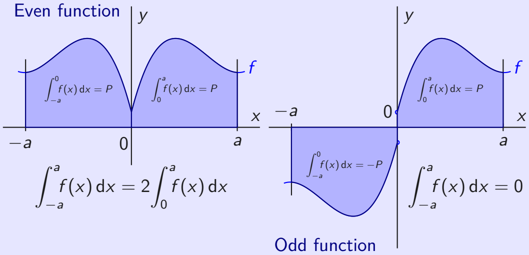
f is even. $\Rightarrow$ • $\displaystyle\int_{-a}^{a} f(x)\,\mathrm{d}x = \int_{-a}^{0} f(x)\,\mathrm{d}x + \int_{0}^{a} f(x)\,\mathrm{d}x = 2\int_{0}^{a} f(x)\,\mathrm{d}x.$

- $\displaystyle\int_{-\pi}^{\pi} \sin|x|\,\mathrm{d}x = \left[\begin{array}{c}\sin|x| \text{ is continuous}\\ \text{and even on } \langle -\pi;\pi\rangle\end{array}\right]$

$$= 2\int_{0}^{\pi} \sin|x|\,\mathrm{d}x = 2\int_{0}^{\pi} \sin x\,\mathrm{d}x$$

$$= 2\Big[-\cos x\Big]_{0}^{\pi} = -2\cdot(-1) + 2\cdot 1 = 4.$$

- $\displaystyle\int_{-1}^{1} \frac{x^3\sqrt[3]{x^2+\sin^2 x}}{x^4+1}\,\mathrm{d}x = \left[\begin{array}{c}\text{integrand is continuous}\\ \text{and odd function on } \langle -1;1\rangle\end{array}\right] = 0.$



Integrating even and odd functions

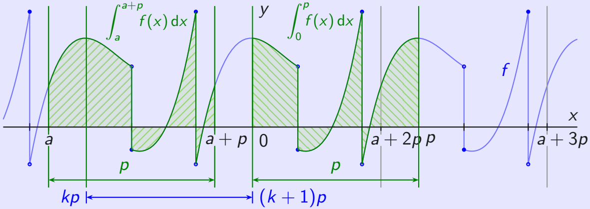
A function $f \in R_{\langle a;b\rangle}$ is periodic with period $p > 0$, $f(x) = f(x + kp)$ for all $x \in \langle a; b\rangle$, $k \in Z$.

If we substituting $x = \varphi(t) = t - kp$, then $t = x + kp$, $t \in \langle a + kp; b + kp\rangle$, $\mathrm{d}t = \mathrm{d}x$, $f(x + kp) \in R_{\langle a+kp;b+kp\rangle}$ and holds:

- $$\int_a^b f(x)\,\mathrm{d}x = \begin{bmatrix} x = b \mapsto t = b + kp \\ x = a \mapsto t = a + kp \end{bmatrix} = \int_{a+kp}^{b+kp} f(t + kp)\,\mathrm{d}t = \int_{a+kp}^{b+kp} f(t)\,\mathrm{d}t = \int_{a+kp}^{b+kp} f(x)\,\mathrm{d}x.$$

A function $y = f(x)$ is periodic with period $p > 0$, a real point $a \in R$, then holds:

- $f \in R_{\langle 0;p\rangle}$ $\Leftrightarrow$
- $f \in R_{\langle a;a+p\rangle}$ and (if they exist)
- $$\int_0^p f(x)\,\mathrm{d}x = \int_a^{a+p} f(x)\,\mathrm{d}x.$$



Thanks for your attention.

Mathematical Analysis supported by wxMaxima

beerb@frcatel.fri.uniza.sk