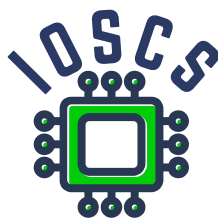


Mendel University in Brno

Mathematical Analysis
supported by wxMaxima
Study text

Rudolf Blaško
Žilinská univerzita v Žiline

Project: Innovative Open Source Courses
for Computer Science Curriculum



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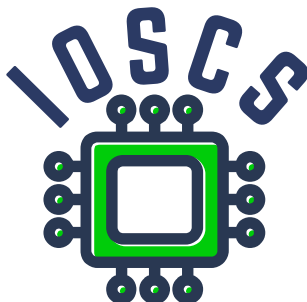
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Preface

The mentioned textbook is one of the basic study materials for the course Mathematical analysis supported by wxMaxima. This textbook was written as one of the outputs of the project “Innovative Open Source Courses for Computer Science”, funded by the Erasmus+ grant no. 2019-1-PL01-KA203-065564. The project is coordinated by West Pomeranian University of Technology in Szczecin (Poland) and is implemented in partnership with Mendel University in Brno (Czech Republic) and University of Žilina (Slovak Republic). The project implementation timeline is September 2019 to December 2022.

The course is focused on the practical teaching of Mathematical Analysis using computing technology (mainly tools for symbolic calculations). Due to the orientation of the project not on commercial programs, the Open Source tool wxMaxima (<https://wxmaxima-developers.github.io/wxmaxima/>) was chosen as a resource and teaching tool. wxMaxima is a high-quality, easy-to-use tool for symbolic as well as numerical calculations. Together with another Open Source tool, Gnuplot, it provides a wide range of 2D and 3D visualization.

The textbook is divided into five chapters. The first chapter is dedicated to the basics of working in the wxMaxima program. There is a brief description of the wxMaxima environment, working with menus, and working at the command level. Here are described basic commands for work, commands for working with constants, commands for creating functions, commands for creating graphs, commands for calculations of limits, derivatives, integrals, for calculations of sums of series, etc. All commands are illustrated with practical demonstrations.

The remaining four chapters are dedicated to mathematical analysis and its teaching using wxMaxima. Some concepts and tasks are briefly described in these chapters from the theory of real functions, number sequences and number series, from differential calculus aimed at investigating the course of a function and from the integral calculus focused on the bases of the indefinite and definite integral. Subsequently, these knowledges are illustrated or solved using wxMaxima tools. All practical examples are verified in the wxMaxima 21.05.2 environment.

Since no one is perfect, you can send any detected errors and shortcomings, as well as suggestions for further improvement of the textbook, to the author’s e-mail address beerb@frcatel.fri.uniza.sk.

Author

Chapter 1

Introduction to wxMaxima

Basic terms


wxMaxima is a document based interface for the computer algebra system Maxima. wxMaxima provides menus and dialogs for many common maxima commands, autocompletion, inline plots and simple animations. wxMaxima is distributed under the GPL license.

xMaxima is a graphical interface for Maxima, written in Tcl/Tk. It also provides the Open Source plotting program for Maxima, which can do some of the plots done by Maxima's default plotter (gnuplot) and a few more that gnuplot cannot do.

Maxima is one of the Open Source programs with open source code. The program can be compiled in various OS, including Windows, GNU/Linux and MacOS X. A precompiled program for GNU/Linux and Windows is available free of charge on the SourceForge website <https://sourceforge.net/projects/maxima/files/>.

After starting the wxMaxima environment, a menu window will appear on the screen at the top. Below the menu is a space where we can enter commands and where outputs appear.

```
(%i1) First input line.  
(%o1) First output line.  
(%i2) Second input line.  
(%o2) Second output line.
```

We enter commands on separate lines (input lines), their execution is ensured by simultaneously pressing the `Shift` keys and `Enter` or by clicking on in the menu icon  (Send the current cell to maxima). Input lines are listed with `(%i1)` and output lines are listed with `(%o1)`. The numbers for the input line and the corresponding output line are identical and based on this number, we can refer to the content of these lines.

```
(%i1) solve(0=x+2,x);  
(%o1) [x = -2]
```

```
(%i2) %i1;
(%o2) solve(0 = x + 2, x)
(%i3) %o1;
(%o3) [x = -2]
```

The commands are executed on new separate lines (output lines). Commands on input lines can be terminated with the symbol `;` (which the system will automatically complete) or the `$` symbol, which suppresses the display of the corresponding output.

We can enter more commands on the input line, but we must separate them symbols `;` or `$`. We can also structure the command on multiple input lines.

```
(%i1) a:2;b:3;solve(a*x+b*x^2=0,x)
(a) 2
(b) 3
(%o1) [x = -2/3, x = 0]
(%i2) a:2$ b:3$ solve(a*x+b*x^2=0,x);
(%o2) [x = -2/3, x = 0]
(%i3) a:2$
      b:3$
      solve(a*x+b*x^2=0,x);
(%o3) [x = -2/3, x = 0]
```

We can save the output in various shapes and then use it in other programs (L^AT_EX, MSWord equation editor, ...). Output `(%o3)` from the previous window we can:

- copy `[Ctrl C]` and `[Ctrl V]`, respectively copy as text (can be used eg. for MSWord equation editor): $x = -2/3, x = 0$,
- copy as L^AT_EX `\[x = -\frac{2}{3}\operatorname{, }x = 0\]`,
- copy as MathML, image, RTF, SVG...

The wxMaxima environment has a well-designed user help, which can be found in the `[Help]` menu. You can also open Help by pressing the `[F1]` key. The manual can be also find on the website https://maxima.sourceforge.io/docs/manual/maxima_369.html.

Basic Commands

The command `apropos` we can find out the exact name of the command using part of its name.

```
(%i1) apropos("plot")
(%o1) [barsplot,boxplot,contour_plot,get_plot_option,gnuplot,...]
```

Command `describe` prints a description of the entered command.

```
(%i1) describe(plot2d);
-- Function: plot2d
plot2d (<expr><,<range_x><,<options><)
plot2d (<expr_<>=<expr_<>,<range_x><,<range_y><,<options><)
plot2d ([parametric,<expr_x><,<expr_<y,<range><],<options><)
plot2d ([discrete,<points><],<options><)
plot2d ([contour,<expr><],<range_x><,<range_y><,<options><)
plot2d ([<type_<>,...,<type_n><],<options><)
There are 5 types of plots that can be plotted by 'plot2d':
    1. Explicit functions. 'plot2d' ...
...
(%o1) true
```

Expressions are entered using the usual characters of operations, sessions and functions. Arguments of functions and commands are in parentheses, multiplication symbol `*` must be entered! The exponentiation is specified by the character `^` or the pair `**`.

Symbol `:` is used to assign a value to the right of the expression to the left. The following commands solve the equation $2x + 3x^2 = 0$ with unknown variable x .

```
(%i1) a:2$ b:3$ solve(a*x+b*x^2=0,x);
(%o1) [x = -2/3, x = 0]
```

In the menu `View` and submenu `Display equations` we can change display output lines for shapes `in 2D`, `as 1D ASCII` or `as ASCII Art`.

The default display is `in 2D`. You can also change the output settings with the command `set_display`. Setting to shape `in 2D` has argument none.

```
(%i1) x/sqrt(x^2+1);set_display('none)$
(%o1)  $\frac{x}{\sqrt{x^2+1}}$  /* in 2D */
```

Using the `ascii` argument command `set_display` change the display output to the form `as 1D ASCII` and using the `xml` argument to form `as ASCII Art`.

```
(%i1) x/sqrt(x^2+1);set_display('ascii)$
(%o1) x/sqrt(x^2 + 1) /* as 1D ASCII */
(%i2) x/sqrt(x^2+1);set_display('xml)$
```

```

      x
(%o2)  -----      /* as ASCII Art */
      2
      sqrt(x + 1)

```

With the `kill` command we can remove variables with all their assignments and properties from memory.

```

(%i1) kill(a,b) /* removes all bindings from the arguments a,b */
(%i2) kill(all) /* removes all items on all infolists */

```

Working with Numbers and Basic Constants

Maxima can work with real numbers written in numerical or symbolic form. The way of writing real numbers can be set in the menu `Numeric` using the switch `Numeric Output` between numeric and symbolic display. Here we can also choose the method and accuracy of numerical display. The setting of the variable `numer` determines the method of displaying.

By default, 16 digits (including the decimal point) are displayed. The display accuracy is defined by the variable `fpprec` and affects the display using `bfloat`. Output `float` always displays the same. We can increase or decrease the accuracy practically indefinitely. We can change it globally and locally for only one variable or command.

```

(%i1) log(2);
(%o1) log(2)
(%i2) log(2),numer;
(%o2) 0.6931471805599453
(%i3) float(log(2));
(%o3) 0.6931471805599453
(%i4) bfloat(log(2));
(%o4) 6.931471805599453b - 1
(%i5) log(2),bfloat;
(%o5) 6.931471805599453b - 1
(%i6) bfloat(log(2)),fpprec=34;
(%o6) 6.931471805599453094172321214581766b - 1
(%i6) bfloat(log(2)),fpprec=134;
(%o6) 6.9314718055994530941723212145[78digits]102057068573368552023575813b - 1

```


Numeric constants e , π , i (imaginary unit) have the prefix `%`, i.e. `%e`, `%pi`, `%i`. This also applies to constants that are part of or the result of calculations. They also have the prefix `%`.

Maxima has predefined constants `inf`, `minf` for real infinite ∞ , $-\infty$ and `infinity` for complex infinity.

Logical constants `true` and `false` they represent truth and untruth.

```
(%i1) %pi+%i+%e;
(%o1)  $\pi + \%i + \%e$ 
(%i2) [minf, inf];
(%o2)  $[-\infty, \infty]$ 
(%i3) infinity;
(%o3) infinity
```

We do not deal with complex numbers in this course, so we will only mention how they are displayed. By default, complex numbers are entered in algebraic form (`rectform`). They can be converted to trigonometric (exponential) form using the command `polarform`.

```
(%i1) z:1+%i;
(z)  $i + 1$ 
(%i2) polarform(z)+rectform(z);
(%o2)  $\sqrt{2} e^{\frac{i\pi}{4}} + i + 1$ 
```

Assignments and Functions

The `:` operator we use to assign values or expressions to variables. We define functions using the assignment `:=`.

```
(%i1) f(x):=x^2+2*x+3;
(%o1)  $f(x) := x^2 + 2x + 3$ 
(%i6) f(x);f(y);f(x+1);f(-2);f(1);
(%o2)  $x^2 + 2x + 3$ 
(%o3)  $y^2 + 2y + 3$ 
(%o4)  $(x + 1)^2 + 2(x + 1) + 3$ 
(%o5) 3
(%o6) 6
```

Maxima contains many more functions than standard programming languages. These are not only the real functions themselves, but also various functions for their support. The basic

functions include `sign(x)`, `abs(x)`, `floor(x)` (bottom whole of x) `round(x)` (rounded x to the nearest whole number), `truncate(x)` (removes all digits after the decimal point), `ceiling(x)` (upper integer x).

```
(%i2) f(x):=sign(x)$
      print(f(-3.6),f(-3.2),f(-3),f(0),f(3),f(3.2),f(3.6))$
      neg neg neg zero pos pos pos
(%i4) f(x):=abs(x)$
      print(f(-3.6),f(-3.2),f(-3),f(0),f(3),f(3.2),f(3.6))$
      3.6 3.2 3 0 3 3.2 3.6
(%i6) f(x):=floor(x)$
      print(f(-3.6),f(-3.2),f(-3),f(0),f(3),f(3.2),f(3.6))$
      -4 -4 -3 0 3 3 3
(%i8) f(x):=round(x)$
      print(f(-3.6),f(-3.2),f(-3),f(0),f(3),f(3.2),f(3.6))$
      -4 -3 -3 0 3 3 4
(%i10) f(x):=truncate(x)$
      print(f(-3.6),f(-3.2),f(-3),f(0),f(3),f(3.2),f(3.6))$
      -3 -3 -3 0 3 3 3
(%i12) f(x):=ceiling(x)$
      print(f(-3.6),f(-3.2),f(-3),f(0),f(3),f(3.2),f(3.6))$
      -3 -3 -3 0 3 4 4
```

We used the command `print` to format the report.

```
(%i3) a:2$ b:log(2),numer$
      print("Logarithm of a number",a," is ",log(a),"=",b)$
      Logarithm of a number 2 is log(2) = 0.6931471805599453
```

Maxima contains many elementary functions. They are, for example `exp(x)=%e^x`, `log(x)`, trigonometric functions, their inverse functions `sin(x)` and `asin(x)`, `cos(x)` and `acos(x)`, `tan(x)` and `atan(x)`, `cot(x)` and `acot(x)`, hyperbolic functions and their inverse functions `sinh(x)` and `asinh(x)`, `cosh(x)` and `acosh(x)`, `tanh(x)` and `atanh(x)`, `coth(x)` and `acoth(x)` etc.

Maxima also includes many features to support them. Some of them are not implemented directly in the wxMaxima environment, but in external libraries called packages. These packages are loaded into the system using the `load` command. We will show the `spang1` package for an example to support work with trigonometric functions.

```
(%i2) print(tan(%pi/8), ratsimp(tan(%pi/8)), trigsimp(tan(%pi/8)))$
      tan( $\frac{\pi}{8}$ ) tan( $\frac{\pi}{8}$ )  $\frac{\sin(\frac{\pi}{8})}{\cos(\frac{\pi}{8})}$ 
(%i3) load(spang1);
(%O3) ../share/trigonometry/spang1.mac
(%i4) tan(%pi/8);
(%O4)  $\sqrt{2} - 1$ 
```

Working with Expressions

Maxima operations and calculations take place in an environment, in which the system presupposes the validity of certain conditions. We may change these terms. Many times we need to change the conditions only locally for a particular calculation without to change global settings. For this purpose, Maxima has a very efficient `ev` command that allows define a specific environment within a single command.

After entering the command `ev(a, b1, b2, ..., bn)` the expression `a` is evaluated if the conditions `b1, b2, ..., bn` are met. These conditions can be equations, assignments, functions, switches (logical settings). The example shows an example of solving a quadratic equation using the command `solve`. Variables `a, b, c` after executing the command `ev` they do not have values assigned.

```
(%i1) ev(solve(a*x^2+b*x+c=0, x), a:2, b:-1, c=-3);
(%O1) [x =  $\frac{3}{2}$ , x = -1]
(%i2) solve(a*x^2+b*x+c=0, x);
(%O2) [x =  $-\frac{\sqrt{b^2-4ac+b}}{2a}$ , x =  $\frac{\sqrt{b^2-4ac-b}}{2a}$ ]
```

Maxima offers several commands for simplifying and editing various expressions. The basic functions can be found in the `Simplify` menu. With the `ratsimp` commands and `trigsimp` we have already met and when adjusting the value of `tan(%pi/8)` they did not have the desired effect.

Maxima offers using the `example` command examples of individual commands. Let's take a look at some of the examples offered by `example(ratsimp)`.

```
(%i2) f(x):=b*(a/b-x)+b*x+a$ print(f(x), "?", ratsimp(f(x)))$
      bx + b( $\frac{a}{b} - x$ ) + a ? 2a
(%i3) ratsimp(a+1/a);
(%O3)  $\frac{a^2+1}{a}$ 
(%i4) ev(x^(a+1/a), ratsimp);
(%O4)  $x^{a+\frac{1}{a}}$ 
```

```
(%i5) ev(x^(a+1/a), ratsimpexpons);
(%o5) x^-a
```

Function `expand` multiplies the relevant members in the expression.

Function `factor` on the contrary, it decomposes the expression.

Function `gfactor` it does so over a field of complex numbers.

```
(%i1) f(x):=(x+1)*(x^2-4)*(x^2+4)$
(%i3) ratsimp(f(x));expand(f(x));
(%o2) x^5 + x^4 - 16x - 16
(%o3) x^5 + x^4 - 16x - 16
(%i6) factor(f(x));gfactor(f(x));factor(100);
(%o4) (x - 2)(x + 1)(x + 2)(x^2 + 4)
(%o5) (x - 2)(x + 1)(x + 2)(x - 2%i)(x + 2%i)
(%o6) 2^2 5^2
```

We decompose a rational polynomial function into partial fractions using the command `partfrac`.

```
(%i1) partfrac((x+1)/(x^2-2*x+1), x);
(%o1) 1/(x-1) + 2/(x-1)^2
```

We can substitute expressions using the commands `subst(a, b, c)` and `ratsubst(a, b, c)`. The expression `a` will be replaced by `b` and subsequently substituted into the expression `c`. When using the `subst` command must be `b` the simplest part (atom) or a complete subexpression of the expression `c`. In the example, the subexpression is not `x+y` complete (missing `z`). The `ratsubst` command it also modifies the resulting expression.

```
(%i2) subst(x+y, a, a^2+b^2); ratsubst(x+y, a, a^2+b^2);
(%o1) (y + x)^2 + b^2
(%o2) y^2 + 2xy + x^2 + b^2
(%i4) subst(a, x+y, x+y+z); ratsubst(a, x+y, x+y+z);
(%o3) z + y + x
(%o4) z + a
```

Limits and Derivatives

In the `Calculus` menu we find functions for solving basic problems of mathematical analysis (limits, derivation, integration, sums of series, decomposition of a function into a Taylor polynomial, ...).

We calculate the limits using the command `limit`. The last parameter determines the direction of unilateral limits, has the values `plus` or `minus` and is optional. If not specified, Maxima calculates the limit as complex.

With the commands `limit(f(x), x, a)` and `limit(f(x), x, a, plus)` we calculate the limits $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a^+} f(x)$.

```
(%i4) limit(1/x,x,0);
      limit(1/x,x,0,plus); limit(1/x,x,0,minus); limit(1/x,t,0);
(%o1) infinity
(%o2) ∞
(%o3) -∞
(%o4) 1/x
```

If we use apostrophe ' before the command, the command will not be executed, it will only be displayed.

```
(%i2) limit(((1-n)/(1+3*n))^(1+4*n),n,inf);
      'limit(((1-n)/(1+3*n))^(1+4*n),n,inf);
(%o1) 0
(%o2) lim_{n→∞} ((1-n)/(3n+1))^{4n+1}
```

Derivatives are calculated using the command `diff`. The parameter that determines the order of derivation is optional.

```
(%i4) f(x) := 2*x^4 - 3*x + sin(x);
      print("f' =", diff(f(x), x), "=", diff(f(x), x, 1))$
      print("f'' =", diff(diff(f(x), x), x), "=", diff(f(x), x, 2),
            "=", diff(f(x), x, 1, x, 1))$
      print("f^(10) =", diff(f(x), x, 10), "=", diff(f(x), x, 1, x, 9))$
(%o1) f(x) := 2x^4 - 3x + sin(x)
      f' = cos(x) + 8x^3 - 3 = cos(x) + 8x^3 - 3
      f'' = 24x^2 - sin(x) = 24x^2 - sin(x) = 24x^2 - sin(x)
      f^(10) = -sin(x) = -sin(x)
```

We calculate partial derivatives using the same command.

```
(%i3) g(x,y):=x^3*y^2-1;
      print("g'_x=",diff(g(x,y),x),",", respectively g'_y=",
            diff(g(x,y),y,1))$
      print("g''_(xx)=",diff(g(x,y),x,2),",", respectively g''_(yx)=",
            diff(g(x,y),y,1,x,1))$
(%o1) g(x,y) := x3y2 - 1
      g'_x = 3x2y2, respectively g'_y = 2x3y
      g''_(xx) = 6xy2, respectively g''_(yx) = 6x2y
```

We calculate the Taylor polynomial n th degree using the command `taylor`. You can find this command in the `Calculus` menu and the `Get Series...` submenu. We calculate Taylor series of functions f degree n in the middle c with the command `taylor(f(x),x,c,n)`. Its coefficients are obtained using the command `coeff`. The use of this command depends on the `taylor` command.

```
(%i1) t1:taylor(sin(x),x,0,5); t2:taylor(sin(x),x,-1,4);
(t1)  x -  $\frac{x^3}{6}$  +  $\frac{x^5}{120}$  + ...
(t2)  -sin(1) + cos(1)(x+1) +  $\frac{\sin(1)(x+1)^2}{2}$  -  $\frac{\cos(1)(x+1)^3}{6}$  -  $\frac{\sin(1)(x+1)^4}{24}$  + ...
(%i3) print(coeff(sin(x),x,5)," and ",coeff(t1,x,5)," ",coeff(t2,x,5))$
      0 and  $\frac{1}{120}$   $\frac{\cos(1)}{120}$ 
```

Taylor polynomial of polynomial is again a polynomial, only it is expressed in another form. Practically, only the coordinate system will change in which we express the polynomial. The beginning of the system moves from point 0 to point -1.

In the following example, the Taylor polynomial of a given polynomial is calculated in another way. Command `aylor` gives three points at the end, even if development is closed.

```
(%i1) f(x):=2*x^5-x^4-3*x^3-x+1;
(%o1) f(x) := 2x5 - x4 + (-3)x3 - x + 1
(%i2) tp1:taylor(f(x),x,-1,5);
(tp1)  2 + 4(x+1) - 17(x+1)2 + 21(x+1)3 - 11(x+1)4 + 2(x+1)5 + ...
(%i4) ratsimp(tp1);expand(tp1);
(%o3) 2x5 - x4 - 3x3 - x + 1
(%o4) 2x5 - x4 - 3x3 - x + 1
(%i6) tpx:ratsubst(t,x+1,f(x));subst(x+1,t,tpx);
```

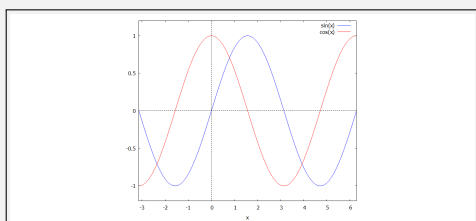
```
(tpx) 2t5 - 11t4 + 21t3 - 17t2 + 4t + 2
(tp2) 2(x + 1)5 - 11(x + 1)4 + 21(x + 1)3 - 17(x + 1)2 + 4(x + 1) + 2
(%i7) tp1-tp2;
(%o7) 0 + ...
```

Function Graphs

We can plot the function graph in several ways. The easiest way is to choose `Plot` in the menu submenu `Plot 2d ...`.

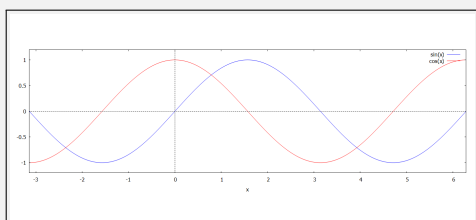
If we choose `Format=gnuplot`, the function is rendered by the command `plot2d` using the Open Source program `Gnuplot` to a new window. `Gnuplot` is automatically installed together with `Maxima`.

```
(%i1) plot2d([sin(x),cos(x)], [x,-%pi,2*%pi], [y,-1.2,1.2],
             [plot_format, gnuplot])$
```



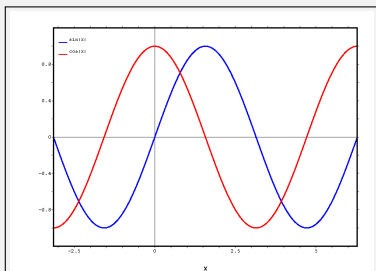
The graphs of the sine and cosine functions were not displayed in the real ratio of the x and y axes, but were optimized for the screen. We have to use e.g. `same_xy` parameter for proper display.

```
(%i1) plot2d([sin(x),cos(x)], [x,-%pi,2*%pi], [y,-1.2,1.2],
             [plot_format, gnuplot],[same_xy])$
```



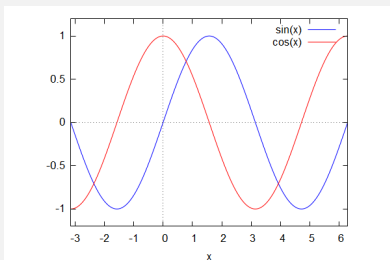
If we choose `Format=wxmaxima`, Maxima will plot the graph using the command `plot2d` to a new window. We can only save the image in postscript.

```
(%i1) plot2d([sin(x),cos(x)], [x,-%pi,2*%pi], [y,-1.2,1.2],
             [plot_format, xmaxima])$
```



If we choose `Format=inline`, Maxima draws a graph using the command `wxplot2d` into your environment.

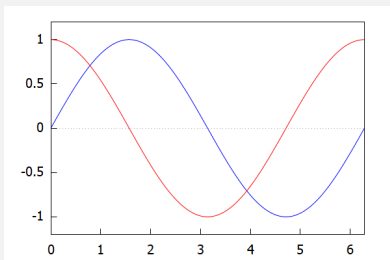
```
(%i1) wxplot2d([sin(x),cos(x)], [x,-%pi,2*%pi], [y,-1.2,1.2])$
```



(%o1)

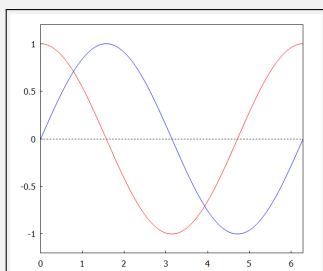
Commands `plot2d` and `wxplot2d` they have the same syntax and have many more parameters. Parameters can be found, for example, with the command `describe(plot2d)`.

```
(%i1) wxdraw2d(xaxis=true,yaxis=true,xrange=[0,2*%pi],yrange=[-1.2,1.2],
               color=blue,explicit((sin(x)),x,0,2*%pi),
               color=red,explicit((cos(x)),x,0,2*%pi))$
```



(%o1)

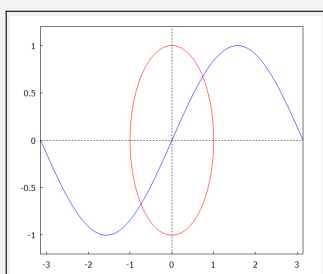

```
(%i1) draw2d(xaxis=true,yaxis=true,xrange=[0,2*%pi],yrange=[-1.2,1.2],
color=blue,explicit((sin(x)),x,0,2*%pi),
color=red,explicit((cos(x)),x,0,2*%pi))$
```



It is better to use the `wxdraw2d` command to print function graphs or `draw2d`, which should be routed to the output of Gnuplot. These commands have a slightly different syntax than the `wxplot2d`, `plot2d`. The print parameters are simpler and clearer. The plotted function must be located in the command `explicit`, `parametric` or `implicit`.

We plot a parametric curve or function in a similar way.

```
(%i1) draw2d(xaxis=true,yaxis=true,xrange=[-%pi,%pi],yrange=[-1.2,1.2],
color=blue,explicit((sin(x)),x,-%pi,%pi),
color=red,nticks=300,parametric(cos(t),sin(t),t,0,2*%pi))$
```



Sequences and Series

Sequences can be created in Maxima, for example, using the command `makelist` or with the statements of the cycle `for..do`.

Command `makelist` creates a list that we can display as a whole and by members.

```
(%i2) S1:makelist(2*n^2-1,n,1,10);S2:makelist(2*n^2-1,n,2,10,2);
(S1) [1, 7, 17, 31, 49, 71, 97, 127, 161, 199]
```

```
(S2) [7, 31, 71, 127, 199]
```

```
(%i4) S1[1];S1[10];
```

```
(%o3) 1
```

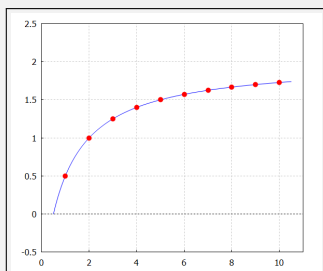
```
(%o4) 199
```

Arranged pairs are enclosed in square brackets and can be displayed as points in a plane. In the following example, a sequence is also generated with its patterns and then plotted with a command `draw2d`.

```
(%i1) S1:makeList([n, (2*n-1)/(n+1)],n,1,10);
```

```
(S1) [[1, 1/2], [2, 1], [3, 5/4], [4, 7/5], [5, 3/2], [6, 11/7], [7, 13/8], [8, 5/3], [9, 17/10], [10, 19/11]]
```

```
(%i2) draw2d(grid=true,xaxis=true,yaxis=true,xrange=[0,11],
yrange=[-0.5,2.5],color=blue,explicit((2*n-1)/(n+1),n,0.5,10.5),
point_type=7,color=red,points(S1))$
```



Using the command `for...do` we will list several members of the sequence $\{2n^2 - 1\}_{n=1}^{\infty}$.

```
(%i1) (for n:1 thru 12 do (a_n: 2*n^2-1, print(a_n)) )$
```

```
1
```

```
7
```

```
17
```

```
31
```

```
49
```

```
71
```

```
97
```

```
127
```

```
161
```

```
199
```

```
241
```

287

A nice example of using the command `for .. do` is a Fibonacci sequence.

```
(%i3) a0:0$ a1:1$
      (for i:1 thru 12 do (an:a1+a0, print(an), a1:a0, a0:an))$
1
1
2
3
5
8
13
21
34
55
89
144
```

We calculate the finite and infinite sum using the command `sum`.

```
(%i1) sum(2*n^2-1,n,1,8);
(%o1) 400
```

With this command, Maxima can calculate the exact sum of some infinite series. The sum of the series can be entered in the menu `Calculus` and the `Calculate Sum...` submenu.

```
(%i2) sum(1/k^2,k,1,inf);
      sum(1/k^2,k,1,inf),simpsum;
```

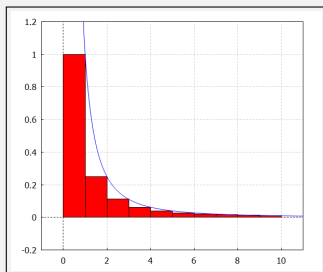
```
(%o1)  $\sum_{k=1}^{\infty} \left(\frac{1}{k^2}\right)$ 
```

```
(%o2)  $\frac{\pi^2}{6}$ 
```

The number series from the previous example can be graphically represented as follows.

```
(%i1) a(n):=1/n^2$ rec:makelist(rectangle([i-1,0],[i,a(i)]),i,1,10)$
```

```
draw2d(grid=true,xaxis=true,yaxis=true,xrange=[-1,11],  
yrange=[-0.2,1.2],border=true,color=black,fill_color=red,rec,  
color=blue,explicit(a(n),n,0,11))$
```



Chapter 2

Real functions

Basic terms

Set is an unordered file (group, aggregate) of objects (things, concepts, numbers, ...), which are called **elements of set**. A set is given if it is determined about each subject whether it belongs or it doesn't. We define a set by expressing all its elements or by rules for its elements:

$$A = \{\text{list of elements}\} = \{x, \text{conditions for } x\}.$$

- $A \subset B$ A is a subset of the set B (each element of the set A also belongs to the set B).
- $A = B$ A and B are equal (sets A and B have the same elements, i.e., $A \subset B, B \subset A$).
- \emptyset , or $\{\}$ An empty set (has no elements).
- $A \neq \emptyset$ A non-empty set A .

- $A \cap B = \{x, x \in A \wedge x \in B\}$ Intersection of sets A and B .
- $A \cup B = \{x, x \in A \vee x \in B\}$ Union of the sets A and B .
- $A - B = \{x, x \in A \wedge x \notin B\}$ Difference of sets A and B .
- $A' = X - A$, where $X \neq \emptyset, A \subset X$ Complement of the set A to the set X .

$$A \times B = \{[x; y], x \in A, y \in B\} \quad \text{Cartesian product of sets } A \text{ and } B (A \times A = A^2).$$

$$A_1 \times A_2 \times \dots \times A_n = \{[x_1; x_2; \dots; x_n], x_1 \in A_1, \dots, x_n \in A_n\} \text{ for } n \in \mathbb{N}$$

Cartesian product of sets $A_1, A_2, \dots, A_n (A \times A \times \dots \times A = A^n)$.

Greek alphabet

α A	alpha	a	η H	eta	é	ν N	nu	n	τ T	tau	t
β B	beta	b	ϑ Θ	theta	th	ξ Ξ	ksi (xi)	x	υ Y	upsilon	y
γ Γ	gamma	g	ι I	iota	i	o O	omicron	o	φ Φ	phi	f
δ Δ	delta	d	κ K	kappa	k	π Π	pi	p	χ X	chi	ch
ε E	epsilon	e	λ Λ	lambda	l	ρ P	rho	r	ψ Ψ	psi	ps
ζ Z	zeta	dz	μ M	mu	m	σ Σ	sigma	s	ω Ω	omega	ó

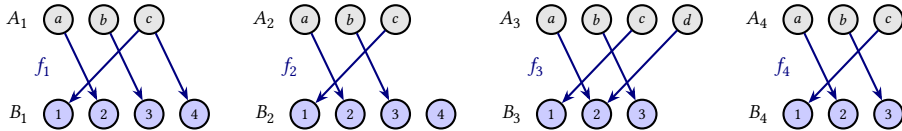
Binary relation f between the sets $A \neq \emptyset$ and $B \neq \emptyset$ is every $f \subset A \times B$. If for each $x \in A$ there is at most one $y \in B$ such that $[x; y] \in f$, then function f is called **function (map, mapping, transformation)** from set A to set B , label $f: A \rightarrow B$. We also write as $[x; y] \in f$ or $y = f(x)$. Relations and functions are sets of ordered pairs.

$x \in A$ Pattern, independent variable, input value, argument.
 $y \in B$ Image, range, dependent variable, output value, value of the function.
 $D(f) = \{x \in A, \exists y \in B : [x; y] \in f\}$ Domain of the function f (set of patterns).
 $H(f) = \{y \in B, \exists x \in D(f) : [x; y] \in f\}$ Codomain of values of the function f (set of images).

$\forall x_1, x_2 \in A: x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$ f is an injection
 (different patterns have different images).
 $\forall y \in B \exists x \in A: y = f(x)$ f is a surjection (every image has a pattern).

 f is a bijection (injection and surjection).

$f = g$ represents the equivalence of $[x; y] \in f \iff [x; y] \in g$,
 i.e. $D(f) = D(g)$ and $f(x) = g(x)$ for all $x \in D(f)$.



$f_1 = \{[a; 2], [b; 3], [c; 1], [c; 4]\}$ is not a function (is a relation),
 $f_2 = \{[a; 2], [b; 3], [c; 1]\}$: $A_2 \rightarrow B_2$ is an injection,
 $f_3 = \{[a; 2], [b; 3], [c; 1], [d; 2]\}$: $A_3 \rightarrow B_3$ is a surjection, $f_4 = \{[a; 2], [b; 3], [c; 1]\}$: $A_4 \rightarrow B_4$ is a bijection.

Example.

Let us denote $f(x) = \sqrt{x}$. Then:

$f : \langle 0; \infty \rangle \rightarrow \langle 0; \infty \rangle$ is a bijection.

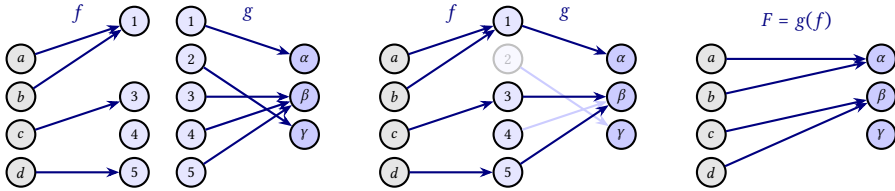
$f : \mathbb{R} \rightarrow \mathbb{R}$ is neither an injection nor a surjection
 ($x = -1$ has no image, $y = -1$ has no pattern).

$f : \langle 0; 4 \rangle \rightarrow \langle 0; 1 \rangle$ is not an injection, it is a surjection ($x = 3$ has no image).

$f : \langle 0; 2 \rangle \rightarrow \langle 0; 4 \rangle$ is an injection, not a surjection ($y = 3$ has no pattern).

$f : \langle 0; 4 \rangle \rightarrow \langle 0; 2 \rangle$ is a bijection. ■

Let $f: A \rightarrow B, g: C \rightarrow D, H(f) \subset C$. Function $F = g(f): A \rightarrow D$ which assigns a value to each $x \in A, z = g(y) = g(f(x)) \in D$, where $y = f(x)$, is called **function composition (composition)** of functions f and g . The function f is called the inner component, the function g is called the outer component.



$$f = \{[a; 1], [b; 1], [c; 3], [d; 5]\}, g = \{[1; \alpha], [2; \gamma], [3; \beta], [4; \beta], [5; \beta]\},$$

$$\text{composition } F = g(f) = \{[a; \alpha], [b; \alpha], [c; \beta], [d; \beta]\}.$$

Example.

Let $g = \{[p; x], [q; z], [r; y], [s; y], [t; y]\}$, $f = \{[a; p], [b; p], [c; r], [d; t]\}$. \Rightarrow

$$F(a) = g[f(a)] = g(p) = x, F(b) = g[f(b)] = g(p) = x, F(c) = g[f(c)] = g(r) = y,$$

$$F(d) = g[f(d)] = g(t) = y \Rightarrow F = g[f] = \{[a; x], [b; x], [c; y], [d; y]\}. \blacksquare$$

Example.

If $f(x) = x^3 : \mathbb{R} \rightarrow \mathbb{R}$, $g(x) = \sin x : \mathbb{R} \rightarrow \langle -1; 1 \rangle$, then

$$f[g(x)] = [g(x)]^3 = \sin^3 x : \mathbb{R} \rightarrow \langle -1; 1 \rangle, g[f(x)] = \sin f(x) = \sin x^3 : \mathbb{R} \rightarrow \langle -1; 1 \rangle. \blacksquare$$

$f: A \rightarrow B, C \subset A$, then the mapping $h: C \rightarrow B$ such that $f(x) = h(x)$ holds for all $x \in C$ is called **restriction f to the set C** , label $h = f|_C$.

$f: A \rightarrow B$ is a bijection. Representation of $g: B \rightarrow A$ such that $[y; x] \in g \Leftrightarrow [x; y] \in f$, i.e. $x = g(y) \Leftrightarrow y = f(x)$, is called the **inverse function to f** , label $g = f^{-1}$.

The set A is **equivalent** to the set B , if there is a bijection $f: A \rightarrow B$, label $A \sim B$.

$A = \emptyset$	The set A is empty.	} The set A is finite.
$A \sim N_n = \{1, 2, \dots, n\}, n \in \mathbb{N}$	The set A is countable finite.	
$A \sim \mathbb{N}$	The set A is countable infinite.	} The set A is infinite.
$A \neq \emptyset$ and $A \not\sim N_n$ and $A \not\sim \mathbb{N}$	The set A is uncountable.	
$A = \emptyset$ or $A \sim N_n$ or $A \sim \mathbb{N}$	The set A is countable.	

Example.

- $\{1, 2, 3\} \sim \{a, b, c\}$, because the function $\{[1; a], [2; b], [3; c]\}$ is a bijection.
- $\mathbb{N} \sim \mathbb{Z}$ because the map $f : \mathbb{N} \rightarrow \mathbb{Z}$ defined by the relation $f(n) = \frac{n}{2}$ for n even and by the relation $f(n) = \frac{1-n}{2}$ for n odd is a bijection.
- $\mathbb{N} \not\sim \mathbb{R}$ because there is no bijection $\mathbb{N} \rightarrow \mathbb{R}$.
- $(-\pi; \pi) \sim \mathbb{R}$ because the function $f(x) = 2 \tan x : (-\pi; \pi) \rightarrow \mathbb{R}$ is a bijection. \blacksquare

$N = \{1, 2, 3, \dots, n, n+1, \dots\}$	Natural numbers.
$Z = \{m - n, m, n \in \mathbb{N}\} = \{0, \pm 1, \pm 2, \dots, \pm n, \dots\}$	Integer numbers.
$Q = \{\frac{m}{n}, m \in \mathbb{Z}, n \in \mathbb{N}\}$	Rational numbers.
$I = \mathbb{R} - Q$	Irrational numbers.

$$R = (-\infty; \infty)$$

Real numbers.

The sum, difference, product and quotient of two rational numbers (with a non-zero denominator) is again a rational number. A rational number (fraction) can have several different expressions. The sum, difference, product and quotient of two irrational numbers it can be irrational as well as rational.

Example.

- $0, 5 = \frac{3}{6} = \frac{1}{2} = \frac{0.7}{1.4}$ and ultimately $\frac{1}{2} = \frac{\sqrt{3}}{2\sqrt{3}}$.
- $\sqrt{2} \cdot \sqrt{2} = 2 \in Q, \sqrt{2} \cdot \sqrt{3} = \sqrt{6} \in I, \frac{\sqrt{8}}{\sqrt{2}} = \sqrt{4} = 2 \in Q, \frac{\sqrt{6}}{\sqrt{2}} = \sqrt{3} \in I.$ ■

$$\left. \begin{array}{l} f, D(f) = N \\ f = \{[n; f(n)], n \in N\} \end{array} \right\} \text{Sequence, for } n \in N \text{ denote } a_n = f(n),$$

$$\text{i.e. } f = \{a_1, a_2, a_3, \dots, a_n, \dots\} = \{a_n\}_{n=1}^{\infty}.$$

$f \sim N$ The sequence f is countable (infinite).

$a_n \in f$ A member of the sequence, represents $[n; f(n)],$

i.e. pattern (order n) and image $a_n = f(n).$

The set R is infinite, but all its elements, i.e. numbers are finite (the number of elements of the set R cannot be expressed by a number). Therefore, we expand the set R by elements $-\infty$ (minus infinity) and ∞ (infinity), label $R^* = R \cup \{-\infty, \infty\}$ (extended set of real numbers).

In the set R^* :

$$\infty + \infty = \infty, a \pm \infty = \pm \infty, \infty \cdot \infty = \infty, a \cdot \infty = \text{sgn } a \cdot \infty, \frac{\infty}{a} = \text{sgn } a \cdot \infty \text{ for } a \neq 0, \frac{a}{\infty} = 0 \text{ for } a \in R.$$

We do not define $\infty - \infty, \infty \cdot 0, \frac{\infty}{\infty}, \frac{\infty}{0}, \frac{a}{0}$ for $a \in R.$ (Indefinite expressions are resolved using limits.)

$$\text{Absolute value of number } a \in R: |a| = \max\{-a, a\} = \begin{cases} -a & \text{for } a \leq 0, \\ a & \text{for } a \geq 0. \end{cases}$$

$$\text{Signum value of number } a \in R: \text{sgn } a = \begin{cases} -1 & \text{for } a < 0, \\ 0 & \text{for } a = 0, \\ 1 & \text{for } a > 0. \end{cases}$$

Example.

If we denote $a^+ = \max\{a, 0\}, a^- = \max\{-a, 0\}$ for $a \in R,$ then:

$$|a| = a^+ + a^-, \quad a = a^+ - a^-, \quad a^+ = \frac{a+|a|}{2}, \quad a^- = \frac{a-|a|}{2}. \blacksquare$$

Sequences of real numbers

Sequence (of real numbers) is every sequence $\{a_n\}_{n=1}^{\infty},$ where $a_n \in R,$ i.e. function $N \rightarrow R.$

Explicit entry: General expression $a_n = f(n), n \in N.$

Recurring entry: Given a_1 and given $a_n, n \in N$ using the previous members.

Example.

$$\{a_n\}_{n=1}^{\infty} = \{2n - 1\}_{n=1}^{\infty} = \{1, 3, 5, \dots\}.$$

- Explicit entry $a_n = 2n - 1, n \in \mathbb{N}$.
- Recurring entry $a_1 = 1, a_{n+1} = a_n + 2, n \in \mathbb{N}$. ■

The following wxMaxima example shows an example of the sequence.

```
(%i3) a(n):=2*n-1$ S:makelist(a(n),n,1,7);
(S) [1,3,5,7,9,11,13]
(%i4) an:1$ (for n:1 thru 7 do (print(an),an:an+2))$
1
3
5
7
9
11
13
```

Sequence $\{a_n\}_{n=1}^{\infty}, a_n \in \mathbb{R}$, numbers $a, b \in \mathbb{R}$.

$$\underbrace{\forall n \in \mathbb{N}: a \leq a_n}_{\text{a is the lower bound, } \{a_n\}_{n=1}^{\infty} \text{ is bounded from below.}}$$

$$\underbrace{\forall n \in \mathbb{N}: a_n \leq b}_{\text{b is an upper bound, } \{a_n\}_{n=1}^{\infty} \text{ is bounded from above.}}$$

$$\underbrace{\forall n \in \mathbb{N}: a \leq a_n}_{\text{a is the lower bound, } \{a_n\}_{n=1}^{\infty} \text{ is bounded from below and above.}}$$

$\forall n \in \mathbb{N}: a_n < a_{n+1}$	Increasing.	} Strictly monotonic.	} Monotonic sequence $\{a_n\}_{n=1}^{\infty}$.
$a_n > a_{n+1}$	Decreasing.		
$\forall n \in \mathbb{N}: a_n \leq a_{n+1}$	Non-decreasing.	} Monotonic sequence $\{a_n\}_{n=1}^{\infty}$.	
$a_n \geq a_{n+1}$	Non-increasing.		
$a_n = a_{n+1}$	Constant (stationary).		

If $\{k_n\}_{n=1}^{\infty} \subset \mathbb{N}$ is an increasing sequence (of natural numbers, indices),
then $\{a_{k_n}\}_{n=1}^{\infty}$ is called **subsequence (selected sequence)** from $\{a_n\}_{n=1}^{\infty}$.

Example.

$$\{a_n\}_{n=1}^{\infty} = \{2n - 1\}_{n=1}^{\infty} = \{1, 3, 5, 7, 9, 11, 13, \dots\}.$$

Subsequences are for example

- $\{a_{k_n}\}_{n=1}^{\infty} = \{a_{2n}\}_{n=1}^{\infty} = \{a_2, a_4, a_6, \dots\} = \{3, 7, 11, \dots\} = \{4n - 1\}_{n=1}^{\infty}$.
- $\{2n - 1\}_{n=1}^{\infty}$. • $\{2n - 1\}_{n=2}^{\infty}$. • $\{101, 109, 235, 637, \dots\}$. ■

```
(%i2) a(n):=2*n-1$ makelist(a(n),n,1,7);
(%o2) [1,3,5,7,9,11,13]
(%i3) makelist(a(2*n),n,1,7);
```

```
(%o3) [3, 7, 11, 15, 19, 23, 27]
(%i4) makelist(a(2*n), n, 2, 7);
(%o4) [7, 11, 15, 19, 23, 27]
(%i5) print(a(51), a(55), a(118), a(319))$
101 109 235 637
```

For every neighborhood $O(a)$ there are infinitely many members $a_n \in O(a)$,
 then $a \in R^* = R \cup \{\pm\infty\}$ is called **accumulation value** of the sequence $\{a_n\}_{n=1}^\infty$.

The set of all accumulation values of the sequence $\{a_n\}_{n=1}^\infty$ we denote by E .

Each sequence $\{a_n\}_{n=1}^\infty$ has at least one accumulation value.

$\sup E = \limsup_{n \rightarrow \infty} a_n$	Limes superior (upper limit)	} They always exist.
$\inf E = \liminf_{n \rightarrow \infty} a_n$	Limes inferior (lower limit)	
$\sup E = \inf E = \lim_{n \rightarrow \infty} a_n$	Limit $\{a_n\}_{n=1}^\infty$ (set E has a single element).	

If $\lim_{n \rightarrow \infty} a_n$ exists, then it is unique.

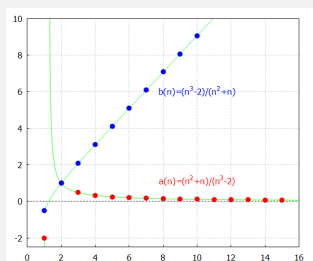
$\exists \lim_{n \rightarrow \infty} a_n = a \in R$	Exists a finite limit, $\{a_n\}_{n=1}^\infty$ converges to the number a , $\{a_n\}_{n=1}^\infty \rightarrow a$.	} $\{a_n\}_{n=1}^\infty$ converges, $\{a_n\}_{n=1}^\infty \rightarrow$.
$\exists \lim_{n \rightarrow \infty} a_n = \pm\infty$	Exists an infinite limit, $\{a_n\}_{n=1}^\infty$ diverges to $\pm\infty$, $\{a_n\}_{n=1}^\infty \rightarrow \pm\infty$.	
$\nexists \lim_{n \rightarrow \infty} a_n$	Do not exists limit, $\{a_n\}_{n=1}^\infty$ oscillates.	} $\{a_n\}_{n=1}^\infty$ diverges, $\{a_n\}_{n=1}^\infty \nrightarrow$.

Changing the finite number (replacement, omission, addition, etc.) of members of the sequence $\{a_n\}_{n=1}^\infty$ does not affect the convergence, or divergence of this sequence.

Example.

- $\lim_{n \rightarrow \infty} \frac{n^2+n}{n^3-2} = \lim_{n \rightarrow \infty} \frac{n^3(n^{-1}+n^{-2})}{n^3(1-2n^{-3})} = \lim_{n \rightarrow \infty} \frac{n^{-1}+n^{-2}}{1-2n^{-3}} = \frac{0+0}{1-0} = 0$.
- $\lim_{n \rightarrow \infty} \frac{n^3-2}{n^2+n} = \lim_{n \rightarrow \infty} \frac{n^2(n-2n^{-2})}{n^2(1+n^{-1})} = \lim_{n \rightarrow \infty} \frac{n-2n^{-2}}{1+n^{-1}} = \frac{\infty-0}{1+0} = \infty$. ■

```
(%i1) a(n):=(n^2+n)/(n^3-2)$ Sa:makelist([n,a(n)],n,1,15)$
      b(n):=(n^3-2)/(n^2+n)$ Sb:makelist([n,b(n)],n,1,15)$
      print("limit a(n)=",limit(a(n),n,inf),"
      limit b(n)=",limit(b(n),n,inf))$
      draw2d(grid=true,xaxis=true,yaxis=true,
             xrange=[0,16],yrange=[-2.5,10],
             color=green,explicit(a(n),n,1,16),point_type=7,
             color=red,points(Sa),
             label(["a(n)=(n^2+n)/(n^3-2)",10,a(10)+1]),
             color=green,explicit(b(n),n,1,16),point_type=7,
             color=blue,points(Sb),
             label(["b(n)=(n^3-2)/(n^2+n)",10,6]))$
(%o1) limit a(n)= 0 limit b(n)= inf
```



- $\{a_n\}_{n=1}^{\infty} \longrightarrow \cdot \quad \Rightarrow \quad \bullet \{a_n\}_{n=1}^{\infty}$ is bounded.
- $\{a_n\}_{n=1}^{\infty}$ is monotonic. $\Rightarrow \bullet \{a_n\}_{n=1}^{\infty} \longrightarrow a \in R^*$.

Example.

- $\lim_{n \rightarrow \infty} \frac{n^4 - n^3}{n^2 + n} = \lim_{n \rightarrow \infty} \frac{n^4(1 - n^{-1})}{n^2(1 + n^{-1})} = \lim_{n \rightarrow \infty} n^2 \cdot \lim_{n \rightarrow \infty} \frac{1 - n^{-1}}{1 + n^{-1}} = \infty \cdot \frac{1 - 0}{1 + 0} = \infty \cdot 1 = \infty.$
- $\lim_{n \rightarrow \infty} \frac{n^2 + n}{n^2 - 2} = \lim_{n \rightarrow \infty} \frac{n^2(1 + n^{-1})}{n^2(1 - 2n^{-2})} = \lim_{n \rightarrow \infty} \frac{1 + n^{-1}}{1 - 2n^{-2}} = \frac{1 + 0}{1 - 0} = 1. \blacksquare$

Example.

$$\lim_{n \rightarrow \infty} n^q = \begin{cases} \infty^q = \infty, & \Rightarrow \bullet \infty \text{ for } q > 0. \\ \lim_{n \rightarrow \infty} n^0 = \lim_{n \rightarrow \infty} 1 = 1. & \Rightarrow \bullet 1 \text{ for } q = 0. \\ \lim_{n \rightarrow \infty} \frac{1}{n^{-q}} = \frac{1}{\lim_{n \rightarrow \infty} n^{-q}} = \frac{1}{\infty} = 0. & \Rightarrow \bullet 0 \text{ for } q < 0 \text{ } (-q > 0). \blacksquare \end{cases}$$

Example (Geometric sequence).

$$L = \lim_{n \rightarrow \infty} q^n \begin{cases} q^n \rightarrow \infty. & \Rightarrow \bullet L = \infty \text{ for } q > 1. \\ q^n = 1^n = 1. & \Rightarrow \bullet L = 1 \text{ for } q = 1. \\ q^n \rightarrow 0. & \Rightarrow \bullet L = 0 \text{ for } q \in (0; 1). \\ q^n = q^{2k} = 1, q^n = q^{2k+1} = -1. & \Rightarrow \bullet \nexists \text{ for } q = -1. \\ q^n = q^{2k} \rightarrow \infty, q^n = q^{2k+1} \rightarrow -\infty. & \Rightarrow \bullet \nexists \text{ for } q < -1. \blacksquare \end{cases}$$

$$\bullet a_n > 0, n \in \mathbb{N}. \Rightarrow \bullet \lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} \text{ (if limits exist).}$$

$$\bullet a_n \geq 0, n \in \mathbb{N}, \lim_{n \rightarrow \infty} \sqrt[n]{a_n} = a \in \mathbb{R}^*. \Rightarrow \bullet \lim_{n \rightarrow \infty} a_n = \begin{cases} 0 & \text{for } a < 1, \\ \infty & \text{for } a > 1. \end{cases}$$

$$\bullet a_n > 0, n \in \mathbb{N}, \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = a \in \mathbb{R}^*. \Rightarrow \bullet \lim_{n \rightarrow \infty} a_n = \begin{cases} 0 & \text{for } a < 1, \\ \infty & \text{for } a > 1. \end{cases}$$

Important limits.

$a, b \in \mathbb{R}, a > 0.$

$$\begin{aligned} \bullet \lim_{n \rightarrow \infty} \sqrt[n]{n!} &= \infty. & \bullet \lim_{n \rightarrow \infty} \sqrt[n]{n} &= 1. & \bullet \lim_{n \rightarrow \infty} \sqrt[n]{a} &= 1. \\ \bullet \lim_{n \rightarrow \infty} \frac{n^n}{n!} &= \infty. & \bullet \lim_{n \rightarrow \infty} \left(1 + \frac{b}{n}\right)^n &= e^b. & \bullet \lim_{n \rightarrow \infty} n(\sqrt[n]{a} - 1) &= \ln a. \end{aligned}$$

The number e is called **Euler's number**. Its value is approximately 2, 718 281 827.

Example.

- $\{0, 1, 0, 1, 0, 1, \dots\}$ has two accumulation values 0 and 1.
- $\{2n - 1\}_{n=1}^{\infty} = \{1, 3, 5, 7, \dots\}$ has one infinite accumulation value ∞ .
- $\{n, -n\}_{n=1}^{\infty} = \{1, -1, 2, -2, \dots\}$ has two accumulation values $\pm\infty$.
- $\left\{n, \frac{1}{n}\right\}_{n=1}^{\infty} = \left\{1, 1, 2, \frac{1}{2}, 3, \frac{1}{3}, \dots\right\}$ has two accumulation values 0 and ∞ .
- $\left\{0, \frac{1}{n}\right\}_{n=1}^{\infty} = \left\{0, 1, 0, \frac{1}{2}, 0, \frac{1}{3}, \dots\right\}$ has one infinite accumulation value 0. \blacksquare

Example.

- Sequences $\{0, 1, 0, 1, \dots\}$, $\{n, -n\}_{n=1}^{\infty}$, $\left\{n, \frac{1}{n}\right\}_{n=1}^{\infty}$ oscillate.
- $\{n^3\}_{n=1}^{\infty} \rightarrow \infty$, i.e. $\lim_{n \rightarrow \infty} n^3 = \infty$,
- $\left\{\frac{1}{n}\right\}_{n=1}^{\infty} \rightarrow 0$, $\left\{0, \frac{1}{n}\right\}_{n=1}^{\infty} \rightarrow 0$, $\{a\}_{n=1}^{\infty} \rightarrow a$, $\{2n - 1\}_{n=1}^{\infty} \rightarrow \infty$, $\{n^n\}_{n=1}^{\infty} \rightarrow \infty$. \blacksquare

Example.

- $\lim_{n \rightarrow \infty} (-1)^n$, $\lim_{n \rightarrow \infty} (-1)^{n+1}$ do not exist, but there are:

$$\lim_{n \rightarrow \infty} [(-1)^n + (-1)^{n+1}] = \lim_{n \rightarrow \infty} 0 = 0, \quad \lim_{n \rightarrow \infty} \frac{(-1)^n}{(-1)^n} = \lim_{n \rightarrow \infty} 1 = 1.$$

- $\lim_{n \rightarrow \infty} n^2 = \lim_{n \rightarrow \infty} n = \infty \implies \lim_{n \rightarrow \infty} (n^2 - n) = \lim_{n \rightarrow \infty} n^2 - \lim_{n \rightarrow \infty} n = \infty - \infty$ makes no sense.

But the limit exists because $\lim_{n \rightarrow \infty} (n^2 - n) = \lim_{n \rightarrow \infty} n \cdot \lim_{n \rightarrow \infty} (n - 1) = \infty \cdot \infty = \infty$. ■

Example.

- $\lim_{n \rightarrow \infty} \frac{n^2+n}{n^3-2} = \lim_{n \rightarrow \infty} \frac{n^3(n^{-1}+n^{-2})}{n^3(1-2n^{-3})} = \lim_{n \rightarrow \infty} \frac{n^{-1}+n^{-2}}{1-2n^{-3}} = \frac{0+0}{1-0} = 0.$
- $\lim_{n \rightarrow \infty} \frac{n^3-2}{n^2+n} = \lim_{n \rightarrow \infty} \frac{n^2(n-2n^{-2})}{n^2(1+n^{-1})} = \lim_{n \rightarrow \infty} \frac{n-2n^{-2}}{1+n^{-1}} = \frac{\infty-0}{1+0} = \infty.$
- $\lim_{n \rightarrow \infty} \frac{n^4-n^3}{n^2+n} = \lim_{n \rightarrow \infty} \frac{n^4(1-n^{-1})}{n^2(1+n^{-1})} = \lim_{n \rightarrow \infty} n^2 \cdot \lim_{n \rightarrow \infty} \frac{1-n^{-1}}{1+n^{-1}} = \infty \cdot \frac{1-0}{1+0} = \infty \cdot 1 = \infty.$
- $\lim_{n \rightarrow \infty} \frac{n^2+n}{n^2-2} = \lim_{n \rightarrow \infty} \frac{n^2(1+n^{-1})}{n^2(1-2n^{-2})} = \lim_{n \rightarrow \infty} \frac{1+n^{-1}}{1-2n^{-2}} = \frac{1+0}{1-0} = 1.$ ■

Example.

- $\lim_{n \rightarrow \infty} [\sqrt{n+1} - n] = \lim_{n \rightarrow \infty} \sqrt{n} [\sqrt{1+n^{-1}} - \sqrt{n}] = \infty \cdot (1 - \infty) = -\infty.$
- $\lim_{n \rightarrow \infty} [\sqrt{n+1} - \sqrt{n}] = \lim_{n \rightarrow \infty} [\sqrt{n+1} - \sqrt{n}] \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} =$
 $= \lim_{n \rightarrow \infty} \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\infty + \infty} = \frac{1}{\infty} = 0.$
- $\lim_{n \rightarrow \infty} [\sqrt{n+1} - n] = \lim_{n \rightarrow \infty} [\sqrt{n+1} - n] \cdot \frac{\sqrt{n+1} + n}{\sqrt{n+1} + n} = \lim_{n \rightarrow \infty} \frac{n+1-n^2}{\sqrt{n+1} + n} =$
 $= \lim_{n \rightarrow \infty} \frac{n}{n} \frac{1+n^{-1}-n}{\sqrt{n^{-1}+n^{-2}}+1} = \frac{1+0-\infty}{0+1} = -\infty.$ ■

Example.

- $\lim_{n \rightarrow \infty} \frac{2^n+3^n}{2^{n+1}+3^{n+1}} = \lim_{n \rightarrow \infty} \frac{3^n (\frac{2}{3})^n+1}{3^n 2(\frac{2}{3})^n+3} = 1 \cdot \frac{0+1}{2 \cdot 0+3} = \frac{1}{3}.$
- $\lim_{n \rightarrow \infty} \left[\frac{n}{2} - \frac{1+\dots+n}{n+2} \right] = \lim_{n \rightarrow \infty} \left[\frac{n}{2} - \frac{n(n+1)}{2(n+2)} \right] = \lim_{n \rightarrow \infty} \frac{n}{2(n+2)} = \lim_{n \rightarrow \infty} \frac{1}{2+\frac{2}{n}} = \frac{1}{2}.$
- $\lim_{n \rightarrow \infty} \sqrt[n]{3^n - 2^n} = \lim_{n \rightarrow \infty} \sqrt[n]{3^n \left[1 - \left(\frac{2}{3}\right)^n \right]} = \lim_{n \rightarrow \infty} 3 \sqrt[n]{1 - \left(\frac{2}{3}\right)^n} = 3 \cdot 1 = 3.$ ■

Example.

Calculate the limit $\{a_n\}_{n=1}^{\infty} = \{0, 2; 0, 23; 0, 233; \dots\}$, i.e. express $0,2\bar{3}$ as a fraction.

$$q = 0,1 < 1 \Rightarrow a_1 = 0,2 \Rightarrow a_2 = 0,2 + 0,03 \Rightarrow a_3 = 0,2 + 0,03 + 0,003 = 0,2 + 0,03(1 + q)$$

$$\Rightarrow a_n = 0,2 + 0,03(1 + q + \dots + q^{n-2}) = 0,2 + 0,03 \frac{1 - q^{n-1}}{1 - q}, n \in \mathbb{N}$$

$$\Rightarrow \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left[0,2 + 0,03 \frac{1 - q^{n-1}}{1 - q} \right]$$

$$= 0,2 + 0,03 \frac{1}{1 - q} = 0,2 + \frac{0,03}{0,9} = \frac{7}{30} \Rightarrow 0,2\bar{3} = \frac{7}{30}. \blacksquare$$

Number series

Number series are closely related to sequences and generalize the concept additions to an infinite number of addends. A simple example is fractions and periodic numbers.

$\{a_n\}_{n=1}^{\infty}$ is a sequence.

$$\Rightarrow \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots + a_n + \dots \text{ is called (infinite number) series.}$$

$$\text{Series } \sum_{n=1}^{\infty} a_n = \underbrace{a_1 + a_2 + a_3 + \dots + a_k}_{s_k = \sum_{i=1}^k a_i \text{ (k-th partial sum)}} + \underbrace{a_{k+1} + a_{k+2} + a_{k+3} + \dots}_{r_k = \sum_{i=k+1}^{\infty} a_i \text{ (k-th rest)}}$$

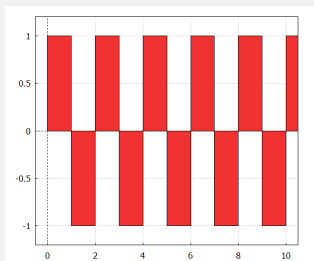
$$\{s_k\}_{k=1}^{\infty} = \{s_1, s_2, s_3, \dots\} = \{s_n\}_{n=1}^{\infty}$$

The sequence of partial sums of the series $\sum_{n=1}^{\infty} a_n$.

The relationship between $\sum_{n=1}^{\infty} a_n$ and the sequence $\{s_n\}_{n=1}^{\infty}$ is mutually exclusive.

- $s_1 = a_1$.
- $s_2 = a_1 + a_2 = s_1 + a_2$.
- $s_3 = a_1 + a_2 + a_3 = s_2 + a_3$.
- ...
- $s_n = a_1 + a_2 + \dots + a_{n-1}$.
- $s_n = a_1 + a_2 + \dots + a_{n-1} + a_n = s_{n-1} + a_n$.
- $a_1 = s_1 - s_0$, where $s_0 = 0$.
- $a_2 = s_2 - s_1$.
- $a_3 = s_3 - s_2$.
- $a_n = s_n - s_{n-1}$, $n \in \mathbb{N}$.

```
(%i1) a(n):=(-1)^(n+1)$ rec:makelist(rectangle([i-1,0],[i,a(i)]),i,1,11)$
draw2d(grid=true,xaxis=true,yaxis=true,
xrange=[-5,10.5],yrange=[-1.2,1.2],
border=true,color=black,fill_color=red,rec)$
```



$\lim_{n \rightarrow \infty} s_n = s \in \mathbb{R}^*$ (if it exists) is called the **sum** of the series $\sum_{n=1}^{\infty} a_n$, label $\sum_{n=1}^{\infty} a_n = s$.

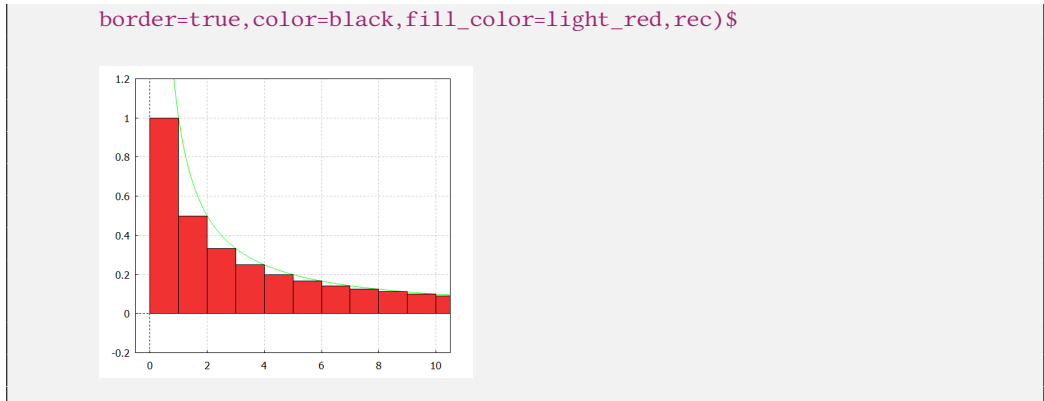
$\exists \lim_{n \rightarrow \infty} s_n = s \in \mathbb{R}$	Exists a finite limit, $\sum_{n=1}^{\infty} a_n$ converges to the sum s , $\sum_{n=1}^{\infty} a_n \rightarrow s$, respectively $\sum_{n=1}^{\infty} a_n = s$.	}	$\sum_{n=1}^{\infty} a_n$ converges, $\sum_{n=1}^{\infty} a_n \rightarrow$.
$\exists \lim_{n \rightarrow \infty} s_n = \pm\infty$	Exists an infinite limit, $\sum_{n=1}^{\infty} a_n$ diverges to $\pm\infty$, $\sum_{n=1}^{\infty} a_n \rightarrow \pm\infty$, respectively $\sum_{n=1}^{\infty} a_n = \pm\infty$.	}	$\sum_{n=1}^{\infty} a_n$ diverges, $\sum_{n=1}^{\infty} a_n \not\rightarrow$.
$\nexists \lim_{n \rightarrow \infty} a_n$	Do not exists limit, $\sum_{n=1}^{\infty} a_n$ oscillates.	}	$\sum_{n=1}^{\infty} a_n \not\rightarrow$.

Changing the finite number (replacement, omission, addition, etc.) of members of series $\sum_{n=1}^{\infty} a_n$ does not affect the convergence, or series divergence. But it has an effect on his sum.

Example (Harmonic series).

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots = \infty. \blacksquare$$

```
(%i1) a(n):=1/n$ rec:makelist(rectangle([i-1,0],[i,a(i)]),i,1,11)$
draw2d(grid=true,xaxis=true,yaxis=true,
xrange=[-.5,10.5],yrange=[-.2,1.2],
color=green,explicit(a(n),n,.5,11),
```



Some rules do not apply for infinite series, e.g. associative law:

$$\sum_{n=1}^{\infty} (-1)^{n+1} = \begin{cases} (1-1) + (1-1) + (1-1) + \dots = 0 + 0 + 0 + \dots = 0, \\ 1 + (-1+1) + (-1+1) + \dots = 1 + 0 + 0 + \dots = 1. \end{cases}$$

Example (Geometric series).

$$\sum_{n=1}^{\infty} q^{n-1} = 1 + q + q^2 + \dots = \frac{1}{1-q} \text{ for all } q \in (-1; 1). \blacksquare$$

$$\sum_{n=1}^{\infty} q^{n-1}, s_n = 1 + q + \dots + q^{n-1} = (1 + q + \dots + q^{n-1}) \frac{1-q}{1-q} = \frac{1-q^n}{1-q} \text{ for } q \neq 1.$$

$$S = \sum_{n=1}^{\infty} q^{n-1} \begin{cases} \lim_{n \rightarrow \infty} \frac{q^n - 1}{q - 1} = \frac{\infty - 1}{q - 1} = \infty. & \Rightarrow \bullet S = \infty \text{ for } q > 1. \\ \lim_{n \rightarrow \infty} \frac{q^n - 1}{q - 1} = \frac{0 - 1}{q - 1} = \frac{1}{1 - q}. & \Rightarrow \bullet S = \frac{1}{1 - q} \text{ for } q \in (-1; 1). \\ -1 + 1 - 1 + 1 - 1 + \dots. & \Rightarrow \bullet \nexists \text{ for } q = -1. \\ \left. \begin{aligned} &= \lim_{n \rightarrow \infty} s_n \left\{ \begin{aligned} \frac{\frac{1}{q} - q^{2k-1}}{\frac{1}{q} - 1} &= \frac{-\frac{1}{q} - \infty}{\frac{1}{q} - 1} = -\infty \text{ for } n = 2k. \\ \frac{\frac{1}{q} - q^{2k+1-1}}{\frac{1}{q} - 1} &= \frac{-\frac{1}{q} + \infty}{\frac{1}{q} - 1} = \infty \text{ for } n = 2k + 1. \end{aligned} \right\} & \Rightarrow \bullet \nexists \text{ for } q < -1. \end{aligned} \right\}$$

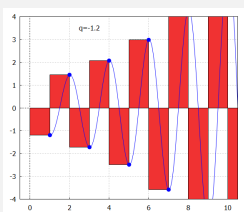
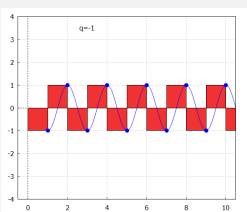
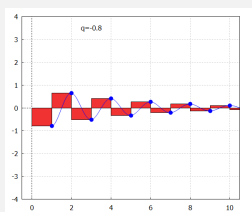
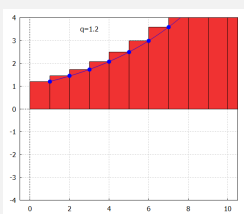
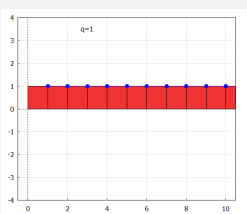
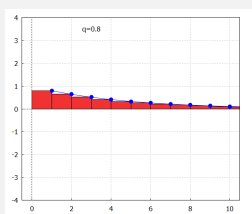
```
(%i4) sq(q):=sum(q^n,n,1,inf)$
      sq(1/2),simpsum; sq(1/3),simpsum; sq(-1/2),simpsum; sq(2),simpsum;
(%o1) 1
```



```
(%02) 1
(%03) 2 1
      2
(%04) sum: sum is divergent.
      #0: sq(q=2)
      -- an error. To debug this try: debugmode(true);
```

It is enough to change the value of q at the beginning in the following example.

```
(%i1) q:0.8$ a(n,q):=q^n$ peca:makelist([i,a(i,q)],i,1,11)$
      reca:makelist(rectangle([i-1,0],[i,a(i,q)]),i,1,11)$
      draw2d(grid=true,xaxis=true,yaxis=true,
      xrange=[-.5,10.5],yrange=[-4,4],
      border=true,color=black,fill_color=light_red,reca,
      label([concat("q=",string(q)),3,3.5]),
      color=blue,explicit(a(n,q),n,1,11),
      point_type=7,color=blue,points(peca))$
```



Necessary convergence condition.

$$\bullet \sum_{n=1}^{\infty} a_n \longrightarrow \implies \bullet \lim_{n \rightarrow \infty} a_n = 0.$$

$\bullet \lim_{n \rightarrow \infty} a_n \neq 0$ does not convergence (the limit does not exist or there is a non-zero).

$$\implies \bullet \sum_{n=1}^{\infty} a_n \not\rightarrow (\text{oscillates or } \longrightarrow \pm\infty).$$

Number series with non-negative terms

The series $\sum_{n=1}^{\infty} a_n$, $a_n \geq 0$, $n \in \mathbb{N}$ (non-negative terms) always has a sum $0 \leq s = \sum_{n=1}^{\infty} a_n \leq \infty$.

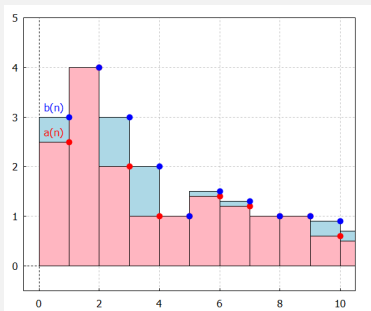
Direct comparison test.

$$0 \leq a_n \leq b_n, n \in \mathbb{N}. \quad \bullet \sum_{n=1}^{\infty} b_n \longrightarrow. \quad \Rightarrow \quad \bullet \sum_{n=1}^{\infty} a_n \longrightarrow.$$

Comparison test (limit form).

$$0 \leq a_n \leq b_n, n \in \mathbb{N}, \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = p \in (0; \infty). \quad \bullet \sum_{n=1}^{\infty} a_n \longrightarrow. \quad \Leftrightarrow \quad \bullet \sum_{n=1}^{\infty} b_n \longrightarrow.$$

```
(%i1) a: [2.5, 4, 2, 1, 1, 1.4, 1.2, 1, 1, 0.6, 0.5]$
pa: makelist([i, a[i]], i, 1, 11)$
ra: makelist(rectangle([i-1, 0], [i, a[i]]), i, 1, 11)$
b: [3.0, 4, 3, 2, 1, 1.5, 1.3, 1, 1, 0.9, 0.7]$
pb: makelist([i, b[i]], i, 1, 11)$
rb: makelist(rectangle([i-1, 0], [i, b[i]]), i, 1, 11)$
draw2d(grid=true, xaxis=true, yaxis=true,
xrange=[-.5, 10.5], yrange=[-.5, 5],
border=true, color=black, fill_color=light_blue,
rb, color=black, fill_color=light_pink, ra,
point_type=7, color=red, points(pa), point_type=7,
color=blue, points(pb),
color=red, label(["a(n)", .5, 2.7]),
color=blue, label(["b(n)", .5, 3.2]))$
```



Ratio test (d'Alembert's ratio test).

$$a_n > 0, n \in \mathbb{N}. \quad \bullet \frac{a_{n+1}}{a_n} \leq q < 1, n \in \mathbb{N}, \text{ where } q \in (0; 1). \quad \Rightarrow \bullet \sum_{n=1}^{\infty} a_n \longrightarrow.$$

$$\bullet 1 \leq \frac{a_{n+1}}{a_n}, n \in \mathbb{N}. \quad \Rightarrow \bullet \sum_{n=1}^{\infty} a_n \longrightarrow \infty.$$

d'Alembert's ratio test (limit form).

$$a_n \geq 0, n \in \mathbb{N}, \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = p. \quad \bullet p < 1. \quad \Rightarrow \bullet \sum_{n=1}^{\infty} a_n \longrightarrow.$$

$$\bullet p > 1. \quad \Rightarrow \bullet \sum_{n=1}^{\infty} a_n \not\rightarrow.$$

for $p = 1$ we cannot decide.

Root test (Cauchy root test).

$$a_n \geq 0, n \in \mathbb{N}. \quad \bullet \sqrt[n]{a_n} \leq q < 1, n \in \mathbb{N}, \text{ where } q \in (0; 1). \quad \Rightarrow \bullet \sum_{n=1}^{\infty} a_n \longrightarrow.$$

$$\bullet 1 \leq \sqrt[n]{a_n}, n \in \mathbb{N}. \quad \Rightarrow \bullet \sum_{n=1}^{\infty} a_n \longrightarrow \infty.$$

Cauchy root test (limit form).

$$a_n > 0, n \in \mathbb{N}, \lim_{n \rightarrow \infty} \sqrt[n]{a_n} = p. \quad \bullet p < 1. \quad \Rightarrow \bullet \sum_{n=1}^{\infty} a_n \longrightarrow.$$

$$\bullet p > 1. \quad \Rightarrow \bullet \sum_{n=1}^{\infty} a_n \not\rightarrow.$$

for $p = 1$ we cannot decide.

Example.

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n^n}{n!} \longrightarrow \infty.$$

$$\bullet \lim_{n \rightarrow \infty} \frac{(n+1)^{n+1}}{(n+1)!} \frac{n!}{n^n} = \lim_{n \rightarrow \infty} \frac{(n+1)^n}{n^n} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e > 1. \quad \Rightarrow \bullet \sum_{n=1}^{\infty} \frac{n^n}{n!} \longrightarrow \infty. \quad \blacksquare$$

Example.

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{a^n}{n!} \longrightarrow \text{for } a > 0.$$

d'Alembert's ratio test:

$$\bullet \lim_{n \rightarrow \infty} \frac{a^{n+1} n!}{(n+1)! a^n} = \lim_{n \rightarrow \infty} \frac{a}{n+1} = \frac{a}{\infty} = 0 < 1. \Rightarrow \bullet \sum_{n=1}^{\infty} \frac{a^n}{n!} \longrightarrow \text{for } a > 0.$$

Cauchy root test:

$$\bullet \lim_{n \rightarrow \infty} \sqrt[n]{\frac{a^n}{n!}} = \lim_{n \rightarrow \infty} \frac{a}{\sqrt[n]{n!}} = \frac{a}{\infty} = 0 < 1. \Rightarrow \bullet \sum_{n=1}^{\infty} \frac{a^n}{n!} \longrightarrow \text{for } a > 0. \blacksquare$$

```
(%i4) an(n):=n^n/n! $ limit(an(n),n,inf,plus);
      limit(an(n+1)/an(n),n,inf,plus);
      limit((an(n))^(1/n),n,inf,plus);
(%o2) ∞
(%o3) e
(%o8) e
(%i9) an(n,a):=a^n/n! $ a:2 $ limit(an(n,a),n,inf,plus);
      limit(an(n+1,a)/an(n,a),n,inf,plus);
      limit((an(n,a))^(1/n),n,inf,plus);
(%o7) 0
(%o8) 0
(%o9) 0
```

Absolute and relative convergence

The series $\sum_{n=1}^{\infty} |a_n|$, $a_n \in \mathbb{R}$, $n \in \mathbb{N}$ always has a sum $0 \leq s = \sum_{n=1}^{\infty} |a_n| \leq \infty$.

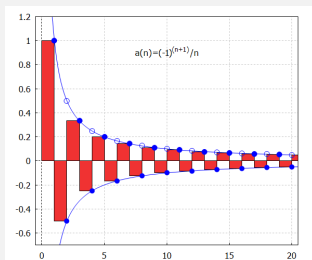
If $\sum_{n=1}^{\infty} |a_n| \longrightarrow$, then $\sum_{n=1}^{\infty} a_n$ **converges absolutely**, label $\sum_{n=1}^{\infty} a_n \xrightarrow{A}$.

If $\sum_{n=1}^{\infty} |a_n| \not\rightarrow$ and $\sum_{n=1}^{\infty} a_n \longrightarrow$, then $\sum_{n=1}^{\infty} a_n$ **converges relatively**, label $\sum_{n=1}^{\infty} a_n \xrightarrow{R}$.

$$\bullet \sum_{n=1}^{\infty} |a_n| \longrightarrow. \Rightarrow \bullet \sum_{n=1}^{\infty} a_n \longrightarrow.$$

```
(%i1) a(n):=(-1)^(n+1)/n $ pa:makelist([i,a(i)],i,1,21)$
      ra:makelist(rectangle([i-1,0],[i,a(i)]),i,1,21)$
      draw2d(grid=true,xaxis=true,yaxis=true,
```

```
xrange=[-.5,20.5],yrange=[-.7,1.2],
color=blue,explicit(abs(a(n)),n,.5,21),
explicit(-abs(a(n)),n,.5,21),
border=true,color=black,fill_color=light_red,ra,
label(["a(n)=(-1)^(n+1)/n",10,.9]),
point_type=6,color=blue,points(abs(pa)),point_type=7,
color=blue,points(pa))$
```



Alternating series test (Leibniz criterion).

- $a_n \geq 0, n \in \mathbb{N}, \{a_n\}_{n=1}^{\infty}$ is non-increasing.
 - $\lim_{n \rightarrow \infty} a_n = 0.$
- $$\left. \vphantom{\begin{matrix} \bullet \\ \bullet \end{matrix}} \right\} \Rightarrow \bullet \sum_{n=1}^{\infty} (-1)^{n+1} a_n \longrightarrow.$$

Example.

(Anharmonic series) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{(-1)^{n+1}}{n} + \dots \xrightarrow{R}.$ ■

Example.

- $\sum_{n=1}^{\infty} 0 = 0$, because $\{s_n\}_{n=1}^{\infty} = \{0\}_{n=1}^{\infty}, \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} 0 = 0.$
- $\sum_{n=1}^{\infty} 1 = \infty$, because $\{s_n\}_{n=1}^{\infty} = \{n\}_{n=1}^{\infty}, \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} n = \infty.$
- $\sum_{n=1}^{\infty} (-1) = -\infty$, because $\{s_n\}_{n=1}^{\infty} = \{-n\}_{n=1}^{\infty}, \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} (-n) = -\infty.$
- $\sum_{n=1}^{\infty} (-1)^n$ oscillates because $\{s_n\}_{n=1}^{\infty} = \{-1, 0, -1, 0, \dots\}$ and $\lim_{n \rightarrow \infty} s_n$ does not exist.
- $\sum_{n=1}^{\infty} \frac{1}{n!} = 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots + \frac{1}{n!} + \dots = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!}\right) = e.$ ■

Example.

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots = 1.$$

$$a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \text{ for all } n \in \mathbb{N}. \Rightarrow$$

$$s_n = a_1 + a_2 + \dots + a_n = \left[\frac{1}{1} - \frac{1}{2} \right] + \left[\frac{1}{2} - \frac{1}{3} \right] + \dots + \left[\frac{1}{n} - \frac{1}{n+1} \right] = 1 - \frac{1}{n+1}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left[1 - \frac{1}{n+1} \right] = 1 - 0 = 1. \blacksquare$$

Example.

- $\sum_{n=1}^{\infty} \frac{1}{n} = \infty, \forall n \in \mathbb{N}: \sqrt{n} \leq n, \text{ i.e. } \frac{1}{n} \leq \frac{1}{\sqrt{n}} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \rightarrow \infty.$

- $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \rightarrow$ (see example), $\forall n \in \mathbb{N}: (n+1)^2 = (n+1)(n+1) \geq n(n+1)$

$$\Rightarrow \frac{1}{(n+1)^2} \leq \frac{1}{n(n+1)} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} \rightarrow \blacksquare$$

Example.

Investigate convergence $\sum_{n=1}^{\infty} a_n = 1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \dots$.

$$\sum_{n=1}^{\infty} a_n = 1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \dots$$

For sequence of partial sums $\{s_n\}_{n=1}^{\infty}$ of the series $\sum_{n=1}^{\infty} a_n$:

$$n \in \mathbb{N} \Rightarrow s_{2n} = 1 - 1 + \frac{1}{2} - \frac{1}{2} + \dots + \frac{1}{n} - \frac{1}{n} = 0, \quad s_{2n+1} = s_{2n} + \frac{1}{n+1} = \frac{1}{n+1} \Rightarrow$$

$$\lim_{n \rightarrow \infty} s_{2n} = \lim_{n \rightarrow \infty} s_{2n+1} = 0 \Rightarrow \lim_{n \rightarrow \infty} s_n = \sum_{n=1}^{\infty} a_n = 0.$$

$$\left. \begin{array}{l} \sum_{n=1}^{\infty} |a_n| = 1 + 1 + \frac{1}{2} + \frac{1}{2} + \dots + \frac{1}{n} + \frac{1}{n} + \dots = 2 \sum_{n=1}^{\infty} \frac{1}{n} = \infty. \end{array} \right\} \Rightarrow \sum_{n=1}^{\infty} a_n \xrightarrow{R} \blacksquare$$

Functions

The function $y = f(x), x \in D(f)$, i.e. $f: D(f) \rightarrow H(f)$.

$D(f) \subset \mathbb{R}$ A function of a real variable.

$H(f) \subset \mathbb{R}$ A real function.

Explicit form:	$y = f(x), x \in D(f)$	(Analytical formula.)
Parametric form:	$f: x = \varphi(t), y = \psi(t), t \in J, J \subset \mathbb{R}$	(Auxiliary functions φ, ψ .)
Implicit form:	$f: F(x, y) = 0$, conditions for $[x; y]$.	(Implicit equation.)

Function $y = f(x), x \in D(f), A \subset D(f)$.

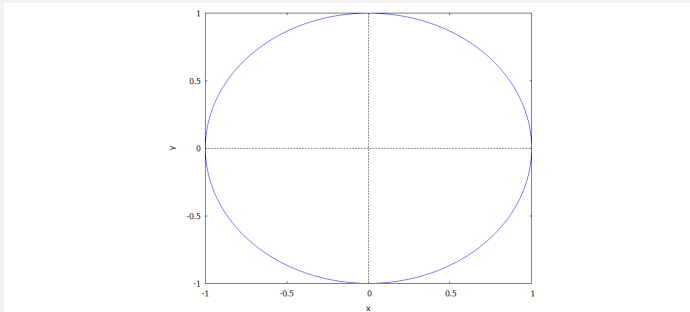
$\forall x_1, x_2 \in A: x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ f is an injection
(different patterns have different images).

$\forall y \in B \exists x \in A: y = f(x)$ f is a surjection
(every image has a pattern), $f(A) = B$.

f is a bijection (injection and surjection).

If we want to display a implicit function, we need to load it `implicit_plot` library.

```
(%i1) load(implicit_plot);
(%o1) ../share/contrib/implicit_plot.lisp
(%i2) implicit_plot(x^2+y^2-1, [x,-1,1], [y,-1,1])$
implicit_plot is now obsolete. Using plot2d instead:
plot2d (y^2+x^2-1=0, [x,-1,1], [y,-1,1])
```

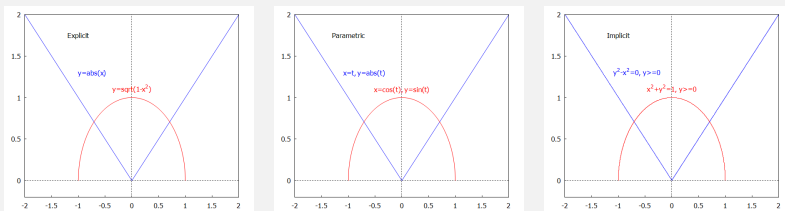


Example.

We can define the function $f: y = |x|, x \in \mathbb{R}$, for example:

- Explicit form: $y = \sqrt{x^2}$, respectively $y = \max\{-x, x\}$.
- Parametric form: $x = t, y = |t|, t \in \mathbb{R}$, respectively $x = t, y = \sqrt{t^2}, t \in \mathbb{R}$.
- Implicit form: $y^2 - x^2 = 0, y \geq 0$, respectively $y - |x| = 0$. ■

```
(%i1) load(implicit_plot)$
(%i2) draw2d(xaxis=true,yaxis=true,xrange=[-2,2],yrange=[-.2,2],
color=blue,explicit(abs(x),x,-2,2),label(["y=abs(x)",-.75,1.3]),
color=red,explicit(sqrt(1-x^2),x,-1,1),
label(["y=sqrt(1-x^2)",0,1.1]),
color=black,label(["Explicit",-1,1.75]))$
(%i3) draw2d(xaxis=true,yaxis=true,xrange=[-2,2],yrange=[-.2,2],
color=blue,parametric(t,abs(t),t,-2,2),
label(["x=t, y=abs(t)",-.7,1.3]),
color=red,nticks=100,parametric(cos(t),sin(t),t,0,%pi),
label(["x=cos(t), y=sin(t)",0,1.1]),
color=black,label(["Parametric",-1,1.75]))$
(%i4) draw2d(xaxis=true,yaxis=true,xrange=[-2,2],yrange=[-.2,2],
color=blue,implicit(y^2-x^2,x,-2,2,y,0,2),
label(["y^2-x^2=0, y>=0",-.65,1.3]),
color=red,implicit(x^2+y^2-1,x,-1,1,y,0,1),
label(["x^2+y^2=1, y>=0",0,1.1]),
color=black,label(["Implicit",-1,1.75]))$
```



Function $y = f(x)$, $x \in D(f)$, $A \subset D(f)$.

$$\left. \begin{array}{l} \forall x \in A: a \leq f(x) \quad a \text{ is the lower bound, } f \text{ is bounded from below} \\ \forall x \in A: f(x) \leq b \quad b \text{ is the upper bound, } f \text{ is bounded from above} \end{array} \right\} \text{ on the set } A.$$

f is bounded (from below and above)

$$\left. \begin{array}{l} \text{not bounded from below on the set } A \quad f \text{ is unbounded from below} \\ \text{not bounded from above on the set } A \quad f \text{ is unbounded from above} \\ \text{not bounded on the set } A \quad f \text{ is unbounded (below or above)} \end{array} \right\} \text{ on the set } A.$$

Function $y = f(x)$, $x \in D(f)$, $A \subset D(f)$.

$A \neq D(f)$. \Rightarrow A local property on the set A .

$A = D(f)$. \Rightarrow Global property (on the entire domain).

Function $y = f(x)$, $x \in D(f)$, $A \subset D(f)$.

$$\begin{array}{ll}
 \inf f(A) = \inf \{f(x), x \in A\} & \text{Local infimum.} \\
 \sup f(A) = \sup \{f(x), x \in A\} & \text{Local supremum.} \\
 \inf f(x) = \inf \{f(x), x \in D(f)\} & \text{Global infimum.} \\
 \sup f(x) = \sup \{f(x), x \in D(f)\} & \text{Global supremum.}
 \end{array}
 \left. \begin{array}{l} \\ \\ \\ \end{array} \right\} \begin{array}{l} \text{on the set } A. \\ \\ \text{(on the definition domain).} \end{array}$$

Function $y = f(x)$, $x \in D(f)$, $A \subset D(f)$, $x_0 \in A$.

$$\begin{array}{ll}
 \forall x \in A: f(x_0) \leq f(x) & \text{Minimum.} \\
 f(x_0) \geq f(x) & \text{Maximum.} \\
 \forall x \in A, x \neq x_0: f(x_0) < f(x) & \text{Strict minimum.} \\
 f(x_0) > f(x) & \text{Strict maximum.}
 \end{array}
 \left. \begin{array}{l} \\ \\ \\ \end{array} \right\} \begin{array}{l} \text{Extrema} \\ \text{Strict extrema.} \end{array}
 \left. \begin{array}{l} \\ \\ \\ \end{array} \right\} \text{on the set } A.$$

$A \neq D(f)$ Local extrema on the set A .

$A = D(f)$ Global (absolute) extrema (on the domain).

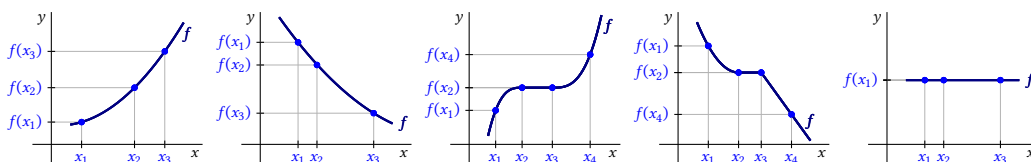
Example.

$$f: y = x^2 + 1, x \in \mathbb{R}.$$

- f is bounded from below, not bounded from above, not bounded.
- The minimum (global) function f is 1, the function f acquires it at the point $x = 0$, the maximum does not exist.
- f is bounded on the interval $\langle -1; 2 \rangle$.
- The local minimum of the function f on the interval $\langle -1; 2 \rangle$ is 1 and acquires it at the point $x = 0$, there is no local maximum, the local supremum is 5. ■

Function $y = f(x)$, $x \in D(f)$, $A \subset D(f)$.

$$\begin{array}{ll}
 \forall x_1, x_2 \in A, x_1 < x_2: f(x_1) < f(x_2) & \text{Increasing.} \\
 f(x_1) > f(x_2) & \text{Descending.} \\
 \forall x_1, x_2 \in A, x_1 < x_2: f(x_1) \leq f(x_2) & \text{Non-decreasing.} \\
 f(x_1) \geq f(x_2) & \text{Non-increasing.} \\
 \forall x_1, x_2 \in A, x_1 < x_2: f(x_1) = f(x_2) & \text{Constant.}
 \end{array}
 \left. \begin{array}{l} \\ \\ \\ \\ \end{array} \right\} \begin{array}{l} \text{Strictly monotonic.} \\ \\ \\ \end{array}
 \left. \begin{array}{l} \\ \\ \\ \end{array} \right\} \begin{array}{l} \text{Monotonic} \\ \text{on the set } A. \end{array}$$



Increasing, decreasing, non-decreasing, non-increasing and constant function (from left to right).

Function $y = f(x)$, $x \in D(f)$.

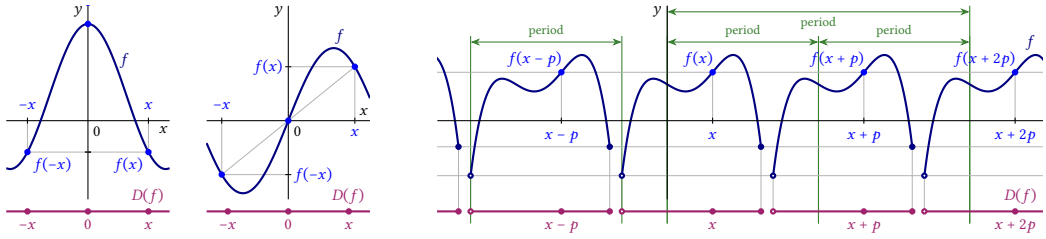
$$\forall x \in D(f): -x \in D(f), f(x) = f(-x) \\
 f(x) = -f(-x)$$

Even function.

Odd function.

$$\forall x \in D(f): x \pm p \in D(f), f(x) = f(x \pm p), p \in \mathbb{R} - \{0\}$$

Periodic function, p is the period.

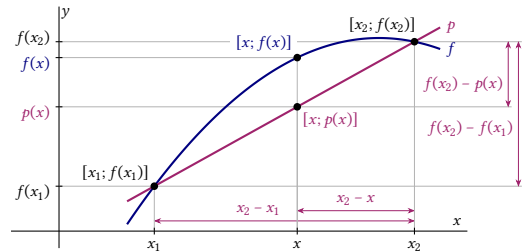
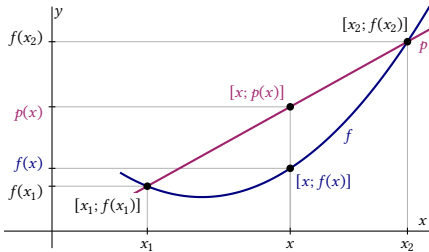


Even function (left), odd function (middle), periodic function (right)

Function $y = f(x)$, $x \in D(f)$, $I \subset D(f)$ is an interval, points $x_1, x_2 \in I$, $x_1 < x_2$.

The line $p(x) = \frac{x_2 - x}{x_2 - x_1} f(x_1) + \frac{x - x_1}{x_2 - x_1} f(x_2)$, $x \in \mathbb{R}$ connects the points $[x_1; f(x_1)]$ and $[x_2; f(x_2)]$.

$\forall x \in I, x_1 < x < x_2:$	$f(x) \leq p(x)$	Convex	} on the interval I .
	$f(x) < p(x)$		
$\forall x \in I, x_1 < x < x_2:$	$f(x) \geq p(x)$	Concave	
	$f(x) > p(x)$	Strictly concave	



Convex function (left), concave function (right)

$x_0 \in D(f)$ is the **inflection point** f (f has an **inflection** at the point x_0), if exists a neighborhood $O_\delta(x_0)$ such that the function f :

- is on $O_\delta^-(x_0) = (x_0 - \delta; x_0)$ strictly convex
 - is on $O_\delta^+(x_0) = (x_0; x_0 + \delta)$ strictly concave
- } respectively { strictly concave.
strictly convex.

Elementary Functions

Elementary function is called each function created using the operations of addition, subtraction, multiplication, division and addition from basic elementary functions

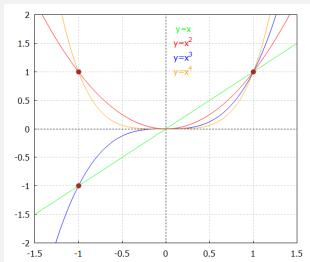
- $y = 1$, • $y = x$, • $y = e^x$, • $y = \ln x$, • $y = \sin x$, • $y = \arcsin x$, • $y = \arctan x$.

A polynomial of degree n is called

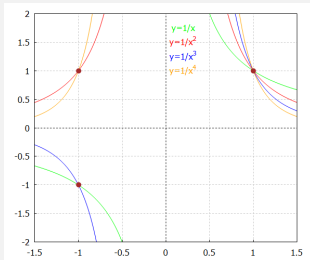
$$f_n: y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n, \text{ where } a_0, a_1, \dots, a_n \in \mathbb{R}, n \in \mathbb{N} \cup \{0\}, a_n \neq 0.$$

- $f_0: y = a_0, a_0 \neq 0$ is called a **constant function**.
- $f_1: y = a_0 + a_1x, a_1 \neq 0$ is called a **linear function**.
- $f_2: y = a_0 + a_1x + a_2x^2, a_2 \neq 0$ is called a **quadratic function**.

```
(%i1) draw2d(grid=true,xaxis=true,yaxis=true,
xrange=[-1.5,1.5],yrange=[-2,2],
color=green,explicit(x,x,-1.5,1.5),label(["y=x",.2,1.75]),
color=red,explicit(x^2,x,-1.5,1.5),label(["y=x^2",.2,1.5]),
color=blue,explicit(x^3,x,-1.5,1.5),label(["y=x^3",.2,1.25]),
color=orange,explicit(x^4,x,-1.5,1.5),label(["y=x^4",.2,1]),
color=brown,point_type=7,points([[ -1,-1],[ -1,1],[ 1,1]]))$
```



```
(%i1) draw2d(grid=true,xaxis=true,yaxis=true,
xrange=[-1.5,1.5],yrange=[-2,2],
color=green,explicit(1/x,x,-1.5,1.5),label(["y=1/x",.2,1.75]),
color=red,explicit(1/x^2,x,-1.5,1.5),label(["y=1/x^2",.2,1.5]),
color=blue,explicit(1/x^3,x,-1.5,1.5),label(["y=1/x^3",.2,1.25]),
color=orange,explicit(1/x^4,x,-1.5,1.5),label(["y=1/x^4",.2,1]),
color=brown,point_type=7,points([[ -1,-1],[ -1,1],[ 1,1]]))$
```



Rational fractional function is called

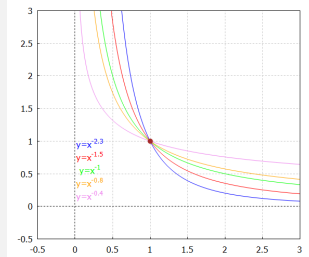
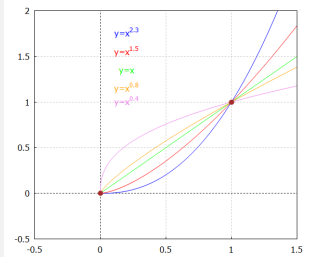
$$f: y = \frac{f_n(x)}{f_m(x)} = \frac{a_0 + a_1x + a_2x^2 + \dots + a_nx^n}{b_0 + b_1x + b_2x^2 + \dots + b_mx^m},$$

where f_n, f_m are polynomials of degrees $n, m \in \mathbb{N} \cup \{0\}$.

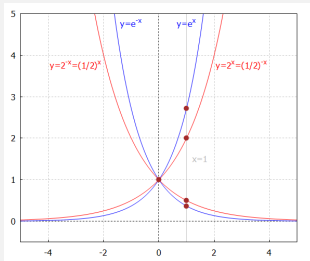
Power function is called

$$f: y = x^r, \text{ where } r \in \mathbb{R}, r \neq 0.$$

```
(%i1) draw2d(grid=true,xaxis=true,yaxis=true,
xrange=[- .5, 1.5],yrange=[- .5, 2],
color=blue,explicit(x^2.3,x,0,1.5),label(["y=x^{2.3}",.2,1.75]),
color=red,explicit(x^1.5,x,0,1.5),label(["y=x^{1.5}",.2,1.55]),
color=green,explicit(x,x,-0,1.5),label(["y=x",.2,1.35]),
color=orange,explicit(x^.8,x,0,1.5),label(["y=x^{0.8}",.2,1.15]),
color=violet,explicit(x^.4,x,0,1.5),label(["y=x^{0.4}",.2,1]),
color=brown,point_type=7,points([[0,0],[1,1]]))$
(%i2) draw2d(grid=true,xaxis=true,yaxis=true,
xrange=[- .5, 3],yrange=[- .5, 3],
color=blue,explicit(x^-2.3,x,0,3),label(["y=x^{-2.3}",.2,.95]),
color=red,explicit(x^-1.5,x,0,3),label(["y=x^{-1.5}",.2,.75]),
color=green,explicit(x^-1,x,-0,3),label(["y=x^{-1}",.2,.55]),
color=orange,explicit(x^- .8,x,0,3),label(["y=x^{-0.8}",.2,.35]),
color=violet,explicit(x^- .4,x,0,3),label(["y=x^{-0.4}",.2,.15]),
color=brown,point_type=7,points([[1,1]]))$
```



```
(%i1) draw2d(grid=true,xaxis=true,yaxis=true,
xrange=[-5, 5],yrange=[- .5, 5],
color=blue,explicit(%e^x,x,-5,5),label(["y=e^x",1,4.75]),
explicit(%e^{-x},x,-5,5),label(["y=e^{-x}",-1,4.75]),
color=red,explicit(2^x,x,-5,5),label(["y=2^x=(1/2)^{-x}",3,3.75]),
explicit(2^{-x},x,-5,5),label(["y=2^{-x}=(1/2)^x",-3,3.75]),
color=grey,parametric(1,t,t,-.5,5),label(["x=1",1.5,1.5]),
color=brown,point_type=7,
points([[0,1],[1,1/2],[1,2],[1,1/%e],[1,%e]]))$
```

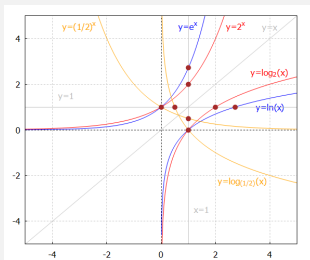


Exponential function with base $a > 0$ is called

$$f: y = a^x, x \in \mathbb{R}.$$

- The most important one is $f: y = \exp x = e^x$ with base e (Euler's number).
- The graph is called the **exponential curve** and passes through the points $[0; 1]$ and $[1; a]$.
- The graphs of the functions $y = a^x$, $y = a^{-x}$ are symmetric along the y axis.

```
(%i1) draw2d(grid=true,xaxis=true,yaxis=true,
xrange=[-5,5],yrange=[-5,5],
color=blue,explicit(exp(x),x,-5,5),label(["y=e^x",1,4.5]),
explicit(log(x),x,.01,5),label(["y=ln(x)",4,1]),
color=red,explicit(2^x,x,-5,5),label(["y=2^x",2.75,4.5]),
explicit(log(x)/log(2),x,.01,5),label(["y=log_2(x)",4,2.5]),
color=orange,explicit((1/2)^x,x,-5,5),label(["y=(1/2)^x",-3,4.5]),
explicit(-log(x)/log(2),x,.01,5),
label(["y=log_{(1/2)}(x)",3,-2.25]),
color=grey,parametric(t,t,t,-5,5),label(["y=x",4,4.5]),
parametric(1,t,t,-5,5),label(["x=1",1.5,-3.5]),
parametric(t,1,t,-5,5),label(["y=1",-3.5,1.5]),
color=brown,point_type=7,
points([[1,0],[0,1],[1,2],[2,1],[1,1/2],[1/2,1],[1,%e],[%e,1]]))$
```



Logarithmic function with base $a > 0$, $a \neq 1$ is called

$$f: y = \log_a x, x \in (0; \infty).$$

• Logarithmic function f is the inverse of the exponential function $y = a^x$, $x \in \mathbb{R}$ with the same base $a > 0$, $a \neq 1$.

• for $x \in (0; \infty)$, $a > 0$, $a \neq 1$ holds $f: y = \log_a x \Leftrightarrow x = a^y$.

• $a > 0$, $a \neq 1$. $\Rightarrow \begin{cases} x = a^{\log_a x} & \text{for } x > 0, \\ x = \log_a a^x & \text{for } x \in \mathbb{R}. \end{cases}$

• The graph is called a **logarithmic curve** and passes through the points $[1; 0]$ and $[a; 1]$.

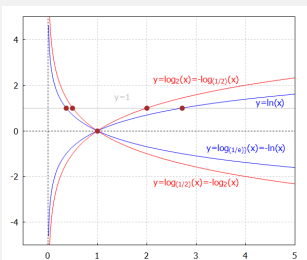
• The graphs of the functions $y = \log_a x$, $y = \log_{a^{-1}} x$ are symmetric along the x axis.

$a = 10$ Decadal logarithm, label $\log x = \log_{10} x$.

$a = e$ Natural logarithm, label $\ln x = \log_e x$.

Exponential function $\exp(x) = e^x$ and logarithmic function $\log(x)$ (natural logarithm) have basics e . If we want to calculate another logarithm, e.g. $\log_2 x$, we have to use construction $\log_2 x = \ln x / \ln 2$.

```
(%i1) draw2d(grid=true,xaxis=true,yaxis=true,
xrange=[-.5,5],yrange=[-5,5],
color=blue,explicit(log(x),x,.01,5),label(["y=ln(x)",4.5,1.25]),
explicit(-log(x),x,.01,5),
label(["y=log_{(1/e)}(x)=-ln(x)",4,-.75]),
color=red,explicit(log(x)/log(2),x,.01,5),
label(["y=log_2(x)=-log_{(1/2)}(x)",3,2.25]),
explicit(-log(x)/log(2),x,.01,5),
label(["y=log_{(1/2)}(x)=-log_2(x)",3,-2.25]),
color=grey,parametric(t,1,t,-.5,5),label(["y=1",1.5,1.5]),
color=brown,point_type=7,
points([[1,0],[1/2,1],[2,1],[1/%e,1],[%e,1]]))$
```



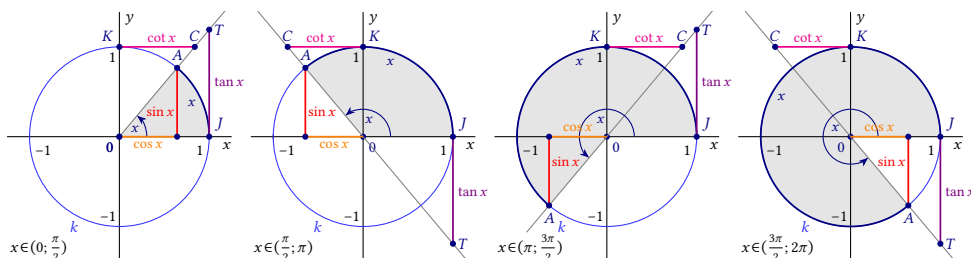
```
(%i1) exp(x)+%e^x;exp(1);
```

```
(%o1) 2% e^x
```

```
(%o2) %e
(%i5) log(x); log(2); log(%e);
(%o3) log(x)
(%o4) log(2)
(%o5) 1
(%i7) log_2(x) := log(x)/log(2); log_2(2);
(%o6) log_2(x) :=  $\frac{\log(x)}{\log(2)}$ 
(%o7) 1
```

Trigonometric (goniometric) functions are:

Sine	$y = \sin x = AA_x :$	$R \rightarrow \langle -1; 1 \rangle.$
Cosine	$y = \cos x = OA_x :$	$R \rightarrow \langle -1; 1 \rangle.$
Tangent	$y = \tan x = \frac{\sin x}{\cos x} = TJ :$	$R - \{\frac{\pi}{2} + k\pi, k \in Z\} \rightarrow R.$
Cotangent	$y = \cot x = \frac{\cos x}{\sin x} = CK :$	$R - \{k\pi, k \in Z\} \rightarrow R.$



Definition of trigonometric functions sine, cosine, tangent and cotangent.

- The number π is called **Ludolf's**. Its value is approximately 3, 141 592 654.
- A circle with a radius $r = 1$ has a circumference of 2π .

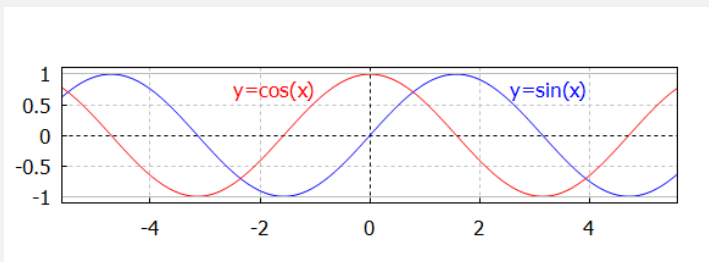
In Maxima, trigonometric functions have the form $\sin(x)$, $\cos(x)$, $\tan(x)$, $\cot(x)$. Arguments of trigonometric functions must be entered in radians. If we want to use degrees, we must first convert to radians.

```
(%i3) tangrad(x) := tan(x/180*%pi); tangrad(22.5); ratsimp(tangrad(22.5));
(%o1) tangrad(x) :=  $\tan\left(\frac{x}{180}\pi\right)$ 
(%o2) tan(0.125%pi)
      rat: replaced 0.125 by 1/8 = 0.125
(%o3) tan( $\frac{\pi}{8}$ )
```

To simplify work with trigonometric functions, we can use commands `trigsimp`, `trigrat`, `trigexpand`, `trigreduce` and packages `atrig1`, `ntrig` and `spangl`, which contain additional support for working with trigonometric functions. We have to load the packages into the system using the command `load`.

```
(%i1) tan(%pi/4);tan(%pi/6);tan(%pi/8);
(%o1) 1
(%o2)  $\frac{1}{\sqrt{3}}$ 
(%o3)  $\tan\left(\frac{\pi}{8}\right)$ 
(%i4) ratsimp(tan(%pi/8));
(%o4)  $\tan\left(\frac{\pi}{8}\right)$ 
(%i5) trigsimp(tan(%pi/8));
(%o5)  $\frac{\sin\left(\frac{\pi}{8}\right)}{\cos\left(\frac{\pi}{8}\right)}$ 
(%i6) load(spangl);
(%o6) ../share/trigonometry/spangl.mac
(%i7) tan(%pi/8);
(%o7)  $\sqrt{2} - 1$ 
```

```
(%i1) draw2d(proportional_axes=xy,grid=true,xaxis=true,yaxis=true,
xrange=[-1.75*%pi-.1,1.75*%pi+.1],yrange=[-1.1,1.1],
color=blue,explicit(sin(x),x,-3*%pi,3*%pi),
label(["y=sin(x)",3.25,.75]),
color=red,explicit(cos(x),x,-3*%pi,3*%pi),
label(["y=cos(x)",-1.75,.75]),
color=grey,parametric(t,1,t,-3*%pi,3*%pi),
parametric(t,-1,t,-3*%pi,3*%pi))$
```



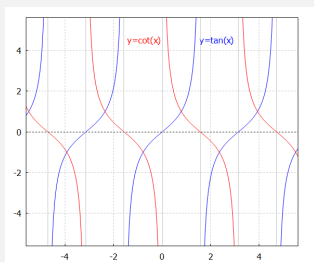
Sum formulas for sine and cosine.

$$x, y \in \mathbb{R}. \quad \bullet \quad \sin(x \pm y) = \sin x \cdot \cos y \pm \cos x \cdot \sin y.$$

$$\bullet \quad \cos(x \pm y) = \cos x \cdot \cos y \mp \sin x \cdot \sin y.$$

$$x \in \mathbb{R}. \quad \bullet \quad \sin^2 x + \cos^2 x = 1.$$

```
(%i1) draw2d(grid=true,xaxis=true,yaxis=true,
xrange=[-1.75*%pi-.1,1.75*%pi+.1],
yrange=[-1.75*%pi-.1,1.75*%pi+.1],
color=blue,explicit(tan(x),x,-3*%pi,3*%pi),
label(["y=tan(x)",2.25,4.5]),
color=red,explicit(cot(x),x,-3*%pi,3*%pi),
label(["y=cot(x)",-.75,4.5]),
color=grey,parametric(0,t,t,-6,6),
parametric(%pi/2,t,t,-6,6),parametric(-%pi/2,t,t,-6,6),
parametric(%pi,t,t,-6,6),parametric(-%pi,t,t,-6,6),
parametric(3*%pi/2,t,t,-6,6),parametric(-3*%pi/2,t,t,-6,6))$
```



Cyclometric functions are inverses of trigonometric functions:

$$\text{Arcsine} \quad y = \arcsin x: \quad \langle -1; 1 \rangle \rightarrow \left\langle \frac{\pi}{2}; \frac{\pi}{2} \right\rangle.$$

$$\text{Arccosine} \quad y = \arccos x: \quad \langle -1; 1 \rangle \rightarrow \langle 0; \pi \rangle.$$

$$\text{Arctangent} \quad y = \arctan x: \quad \mathbb{R} \rightarrow \left(-\frac{\pi}{2}; \frac{\pi}{2} \right).$$

$$\text{Arccotangent} \quad y = \operatorname{arccot} x: \quad \mathbb{R} \rightarrow (0; \pi).$$

There are no inverse functions for trigonometric functions because they are not injective. It is necessary to narrow them appropriately.

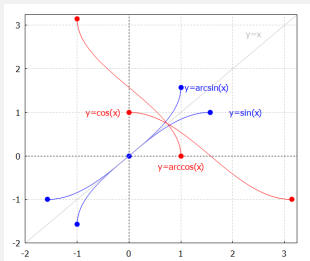
Inverse functions to trigonometric functions have the form $\operatorname{asin}(x)$, $\operatorname{acos}(x)$, $\operatorname{atan}(x)$, $\operatorname{acot}(x)$. At this point we can mention the function $\operatorname{atan2}(x, y)$ defined by the relation $\operatorname{arctan} \frac{x}{y}$.

```
(%i2) asin(1);acos(1);
(%o1)  $\frac{\pi}{2}$ 
(%o2) 0
(%i4) atan2(2,4);atan(1/2);
(%o3)  $\text{atan}\left(\frac{1}{2}\right)$ 
(%o4)  $\text{atan}\left(\frac{1}{2}\right)$ 
```

Sum formulas for cyclometric functions.

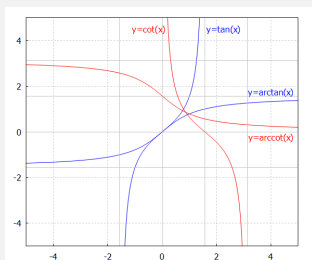
- $\arcsin x + \arccos x = \frac{\pi}{2}$ for $x \in \langle -1; 1 \rangle$.
- $\arctan x + \text{arccot } x = \frac{\pi}{2}$ for $x \in \mathbb{R}$.

```
(%i1) draw2d(grid=true,xaxis=true,yaxis=true,
xrange=[-2,%pi+.1],yrange=[-2,%pi+.1],
color=blue,explicit(sin(x),x,-%pi/2,%pi/2),
label(["y=sin(x)",2.25,1]),
explicit(asin(x),x,-1,1),label(["y=arcsin(x)",1.5,%pi/2]),
point_type=7,points([[0,0],[1,%pi/2],
[-1,-%pi/2],[%pi/2,1],[-%pi/2,-1]]),
color=red,explicit(cos(x),x,0,%pi),label(["y=cos(x)",-.5,1]),
explicit(acos(x),x,-1,1),label(["y=arccos(x)",1,-.25]),
point_type=7,points([[0,1],[1,0],[%pi,-1],[-1,%pi]]),
color=grey,parametric(t,t,t,-5,5),label(["y=x",2.4,2.8]))$
```



```
(%i1) draw2d(grid=true,xaxis=true,yaxis=true,
xrange=[-5,5],yrange=[-5,5],
color=blue,explicit(tan(x),x,-%pi/2+.01,%pi/2-.01),
label(["y=tan(x)",2.25,4.5]),
explicit(atan(x),x,-5,5),label(["y=arctan(x)",4,1.75]),
color=grey,parametric(t,-%pi/2,t,-5,5),parametric(t,%pi/2,t,-5,5),
parametric(-%pi/2,t,t,-5,5),parametric(%pi/2,t,t,-5,5),
color=red,explicit(cot(x),x,.01,%pi-.01),
label(["y=cot(x)",-.5,4.5]),
```

```
explicit(%pi/2-atan(x), x, -5, 5), label(["y=arccot(x)", 4, -.25]),
color=grey, parametric(t, 0, t, -5, 5), parametric(t, %pi, t, -5, 5),
parametric(0, t, t, -5, 5), parametric(%pi, t, t, -5, 5))$
```

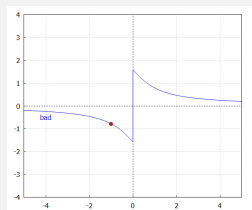
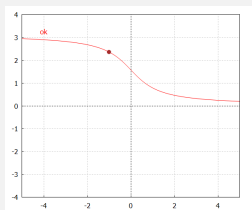


Beware of the inaccurate interpretation of the Arccotangent function in the wxMaxima.

```
(%i4) acot(-1)$ print("acot(-1)=", acot(-1), "is bad")$
%pi/2-atan(-1)$ print("acot(-1)=", %pi/2-atan(-1), "is ok")$
acot(-1) =  $-\frac{\pi}{4}$  is bad
acot(-1) =  $-\frac{3\pi}{4}$  is ok
```

```
(%i5) draw2d(grid=true, xaxis=true, yaxis=true,
xrange=[-5, 5], yrange=[-4, 4],
color=red, explicit(%pi/2-atan(x), x, -5, 5), label(["ok", -4, 3.25]),
color=brown, point_type=7, points([[ -1, %pi/2-atan(-1) ]]))$
```

```
(%i6) draw2d(grid=true, xaxis=true, yaxis=true,
xrange=[-5, 5], yrange=[-4, 4],
color=blue, explicit(acot(x), x, -5, 5), label(["bad", -4, -.5]),
color=brown, point_type=7, points([[ -1, acot(-1) ]]))$
```



Hyperbolic functions are:

$$\text{Hyperbolic sine} \quad y = \sinh x = \frac{e^x - e^{-x}}{2} = \frac{e^{2x} - 1}{2e^x}: \quad R \rightarrow R.$$

$$\text{Hyperbolic cosine} \quad y = \cosh x = \frac{e^x + e^{-x}}{2} = \frac{e^{2x} + 1}{2e^x}: \quad R \rightarrow \langle 1; \infty \rangle.$$

$$\text{Hyperbolic tangent} \quad y = \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}: \quad R \rightarrow (-1; 1).$$

$$\text{Hyperbolic cotangent} \quad y = \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}: \quad R - \{0\} \rightarrow R - \langle -1; 1 \rangle.$$

Hyperbolic functions have similar properties to trigonometric functions, therefore they have similar names.

Sum formulas for hyperbolic sine and hyperbolic cosine.

$$x, y \in R. \quad \bullet \sinh(x \pm y) = \sinh x \cdot \cosh y \pm \cosh x \cdot \sinh y.$$

$$\bullet \cosh(x \pm y) = \cosh x \cdot \cosh y \pm \sinh x \cdot \sinh y.$$

$$x \in R. \quad \bullet \sinh 2x = 2 \sinh x \cdot \cosh x. \quad \bullet \sinh^2 x = \frac{\cosh 2x - 1}{2}.$$

$$\bullet \cosh 2x = \cosh^2 x + \sinh^2 x. \quad \bullet \cosh^2 x = \frac{\cosh 2x + 1}{2}.$$

$$\bullet \cosh^2 x - \sinh^2 x = 1. \quad \bullet \sinh x \pm \cosh x = \pm e^{\pm x}.$$

Moirve formula.

$$\bullet (\cosh x \pm \sinh x)^n = \cosh nx \pm \sinh nx \text{ for } x \in R, n \in N.$$

Hyperbolometric (Inverse hyperbolic) functions are inverses of hyperbolic functions:

• **Inverse hyperbolic sine**

$$y = \operatorname{arsinh} x = \ln(x + \sqrt{x^2 + 1}): \quad R \rightarrow R.$$

• **Inverse hyperbolic cosine**

$$y = \operatorname{arcosh} x = \ln(x + \sqrt{x^2 - 1}): \quad \langle 1; \infty \rangle \rightarrow \langle 0; \infty \rangle.$$

• **Inverse hyperbolic tangent**

$$y = \operatorname{artanh} x = \frac{1}{2} \ln \frac{1+x}{1-x}: \quad (-1; 1) \rightarrow R.$$

• **Inverse hyperbolic cotangent**

$$y = \operatorname{arcoth} x = \frac{1}{2} \ln \frac{x+1}{x-1}: \quad R - \langle -1; 1 \rangle \rightarrow R - \{0\}.$$

Hyperbolic functions are $\sinh(x)$, $\cosh(x)$, $\tanh(x)$, $\coth(x)$ and to them the inverse hyperbolometric functions are $\operatorname{arsinh}(x)$, $\operatorname{arcosh}(x)$, $\operatorname{artanh}(x)$, $\operatorname{arcoth}(x)$.

```
(%i4) sinh(x);cosh(0);tanh(0);coth(1),numer;
(%o1) sinh(x)
(%o2) 1
(%o3) 0
(%o4) 1.313035285499331
(%i8) asinh(x);acosh(1);atanh(0);acoth(1.3),numer;
(%o5) asinh(x)
(%o6) 0
(%o7) 0
(%o8) 1.01844096363052
```

Limit of a function

When investigating a function, it is necessary to characterize its local properties at different intervals and around different important points. The function f does not have to be defined at the point around which we investigate it.

$a \in \mathbb{R}^*$ is an **accumulation point** of the set $A \subset \mathbb{R}$, if for every neighborhood $O(a)$ there exists $x \in O(a) \cap A$, $x \neq a$.

The function f has a **limit** $b \in \mathbb{R}^*$ at the point $a \in \mathbb{R}^*$, label $\lim_{x \rightarrow a} f(x) = b$, if:

- a is the accumulation point of the set $D(f)$.
- For all $\{x_n\}_{n=1}^{\infty} \subset D(f)$, $x_n \neq a$, $\{x_n\}_{n=1}^{\infty} \rightarrow a$ holds $\{f(x_n)\}_{n=1}^{\infty} \rightarrow b$.

The given definition of limits using sequences is called Heine's. We can characterize the limit using the neighborhood $O(a)$ and $O(b)$.

The function f has a **limit** $b \in \mathbb{R}^*$ at the point $a \in \mathbb{R}^*$, label $\lim_{x \rightarrow a} f(x) = b$, if:

- a is the accumulation point of the set $D(f)$.
- For every neighborhood $O(b)$ exists a neighborhood $O(a)$ such that for all $x \in O(a)$, $x \neq a$ holds $f(x) \in O(b)$.

a is the accumulation point of the set $D(f)$.

\Rightarrow There exists (at least one) $\{x_n\}_{n=1}^{\infty} \rightarrow a$, $x_n \in D(f) - \{a\}$
and $\lim_{x \rightarrow a} f(x) = \lim_{n \rightarrow \infty} f(x_n)$ holds.

$$\lim_{x \rightarrow a} f(x) = b. \begin{cases} a \in \mathbb{R}^*. & \begin{cases} a \in \mathbb{R}. & \text{Limit at eigenpoint } a. \\ a = \pm\infty. & \text{Limit at non-eigenpoint } a. \end{cases} \\ b \in \mathbb{R}^*. & \begin{cases} b \in \mathbb{R}. & \text{Finite limit.} \\ b = \pm\infty. & \text{Infinite limit.} \end{cases} \end{cases}$$

- $\lim_{x \rightarrow a} f(x) = b, a \in \mathbb{R}^*, b \in \mathbb{R}. \implies$ • There exists $O(a)$ in which f is bounded.

$a \in \mathbb{R}^*$ is the accumulation point of sets $D(f)$ and $D(g)$, $O(a)$ is the neighborhood.

$$\forall x \in O(a), x \neq a: \bullet f(x) = g(x). \implies \bullet \lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) \text{ if they exist.}$$

$$\bullet f(x) \leq g(x). \implies \bullet \lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x) \text{ if they exist.}$$

$$\forall x \in O(a), x \neq a: \bullet f(x) < g(x). \implies \bullet \lim_{x \rightarrow a} f(x) < \lim_{x \rightarrow a} g(x) \text{ if they exist.}$$

Two Policemen and a Drunk Theorem.

$a \in \mathbb{R}^*$ is the accumulation point of sets $D(f)$, $D(g)$ and $D(h)$, $O(a)$ is the neighborhood.

$$\left. \begin{array}{l} \bullet \forall x \in O(a), x \neq a: h(x) \leq f(x) \leq g(x). \\ \bullet \lim_{x \rightarrow a} h(x) = \lim_{x \rightarrow a} g(x) = b, b \in \mathbb{R}^*. \end{array} \right\} \implies \bullet \text{There exists } \lim_{x \rightarrow a} f(x) = b.$$

Example.

$$\lim_{x \rightarrow \infty} \frac{\sin x}{x} = 0.$$

∞ is the accumulation point of the domain of the function $y = \frac{\sin x}{x}$.

for $x \in \mathbb{R}$ holds $-1 \leq \sin x \leq 1. \implies -\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$ holds for $x > 0$.

$$\implies 0 = -\lim_{x \rightarrow \infty} \frac{1}{x} \leq \lim_{x \rightarrow \infty} \frac{\sin x}{x} \leq \lim_{x \rightarrow \infty} \frac{1}{x} = 0. \implies \lim_{x \rightarrow \infty} \frac{\sin x}{x} = 0. \blacksquare$$

```
(%i1) limit(sin(x)/x,x,inf);
```

```
(%o1) 0
```

```
(%i1) f(x):=sin(x)/x$ for i:1 thru 10 do
      (x:100^i, print(x," ",ev(f(x),numer)))$
```

```
100 -0.005063656411097588
```

```
10000 -3.056143888882521 \cdot 10^{-5}
```

```
1000000 -3.499935021712929 \cdot 10^{-7}
```

```
100000000 9.31639027109726 \cdot 10^{-9}
```

10000000000	$-4.875060250875107 \cdot 10^{-11}$
1000000000000	$-6.112387023768895 \cdot 10^{-13}$
1000000000000000	$-2.094083074964523 \cdot 10^{-15}$
1000000000000000000	$7.796880066069787 \cdot 10^{-17}$
100000000000000000000	$-9.929693207404051 \cdot 10^{-19}$
10000000000000000000000	$-6.452512852657808 \cdot 10^{-21}$

Example.

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$$

Let us denote $f(x) = \frac{\sin x}{x}$, $x \in D(f) = \left(-\frac{\pi}{2}; \frac{\pi}{2}\right) - \{0\}$, 0 is accumulation point of the set $D(f)$.

For all $x \in D(f)$ holds:

$$\left. \begin{array}{l} 0 < x \Rightarrow 0 < \sin x < x < \tan x \Rightarrow \frac{\sin x}{\sin x} < \frac{x}{\sin x} < \frac{\frac{\sin x}{\cos x}}{\sin x} = \frac{1}{\cos x} \\ x < 0 \Rightarrow \tan x < x < \sin x < 0 \Rightarrow \frac{1}{\cos x} = \frac{\frac{\sin x}{\cos x}}{\sin x} > \frac{x}{\sin x} > \frac{\sin x}{\sin x} \end{array} \right\} \Rightarrow 1 < \frac{x}{\sin x} < \frac{1}{\cos x}.$$

$$\Rightarrow 1 = \lim_{x \rightarrow 0} 1 \leq \lim_{x \rightarrow 0} \frac{x}{\sin x} \leq \lim_{x \rightarrow 0} \frac{1}{\cos x} = 1. \Rightarrow \lim_{x \rightarrow 0} \frac{x}{\sin x} = 1. \blacksquare$$

```
(%i1) limit(sin(x)/x,x,0);
(%o1) 1
```

```
(%i1) f(x):=sin(x)/x$
for i:-1 thru -10 step -1 do (x:1/i, print(x," ",ev(f(x),numer)))$
print("Limit")$
for i:10 thru 1 step -1 do (x:1/i, print(x," ",ev(f(x),numer)))$
-1 0.8414709848078965
-1/2 0.958851077208406
-1/3 0.9815840903884566
-1/4 0.9896158370180917
-1/5 0.9933466539753061
-1/6 0.9953767961604901
-1/7 0.9966021085458455
-1/8 0.9973978670818215
-1/9 0.9979436565895768
-1/10 0.9983341664682815
Limit
1/10 0.9983341664682815
1/9 0.9979436565895768
```

$\frac{1}{8}$	0.9973978670818215
$\frac{1}{7}$	0.9966021085458455
$\frac{1}{6}$	0.9953767961604901
$\frac{1}{5}$	0.9933466539753061
$\frac{1}{4}$	0.9896158370180917
$\frac{1}{3}$	0.9815840903884566
$\frac{1}{2}$	0.958851077208406
1	0.8414709848078965

The limit of the composite function.

$y = f(x)$, $y = g(x)$, $H(f) \subset D(g)$, $a, b, c \in \mathbb{R}^*$, $O(a)$ is a neighborhood.

$$\left. \begin{array}{l} \bullet \lim_{x \rightarrow a} f(x) = b, \lim_{u \rightarrow b} g(u) = c. \\ \bullet \forall x \in O(a), x \neq a: f(x) \neq b, \\ \text{respectively } \bullet g(b) = c. \end{array} \right\} \Rightarrow \bullet \lim_{x \rightarrow a} g(f(x)) = \lim_{u \rightarrow b} g(u) = c.$$

$$\text{Substitution } u = f(x). \Rightarrow \lim_{x \rightarrow a} g(f(x)) = \left[\begin{array}{l} \text{Subst. } u = f(x) \\ x \rightarrow a, u \rightarrow b. \end{array} \right] = \lim_{u \rightarrow b} g(u).$$

$\lim_{x \rightarrow a} f(x) = b$, $\lim_{x \rightarrow a} g(x) = c$, $a, b, c \in \mathbb{R}^*$, $r \in \mathbb{R}$. \Rightarrow (If the expressions make sense.)

$$\bullet \lim_{x \rightarrow a} |f(x)| = \left| \lim_{x \rightarrow a} f(x) \right| = |b|. \quad \bullet \lim_{x \rightarrow a} [r \cdot f(x)] = r \cdot \lim_{x \rightarrow a} f(x) = r \cdot b.$$

$$\bullet \lim_{x \rightarrow a} [f(x) \odot g(x)] = \lim_{x \rightarrow a} f(x) \odot \lim_{x \rightarrow a} g(x) = b \odot c,$$

where \odot is the operation $+$, $-$, \cdot , respectively $/$.

If any of the expressions does not make sense,

it does not necessarily mean that there is no limit.

We have to calculate the limit in another way.

$y = f(x)$, $x \in D(f)$, point $a \in \mathbb{R}$.

$$f^-(x) = f(x)|_{D(f) \cap (-\infty; a)} = f(x)|_{\{x \in D(f), x < a\}}$$

Narrowing the function f to the left.

$$f^+(x) = f(x)|_{D(f) \cap (a; \infty)} = f(x)|_{\{x \in D(f), a < x\}}$$

Narrowing the function f to the right.

$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^-} f^-(x) \quad \text{Left limit (from left).}$$

$$\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^+} f^+(x) \quad \text{Right limit (from right).}$$

} One-sided limits
of the function f at the point a .

$$\lim_{x \rightarrow a} f(x)$$

Two-sided limit of the function f at the point a .

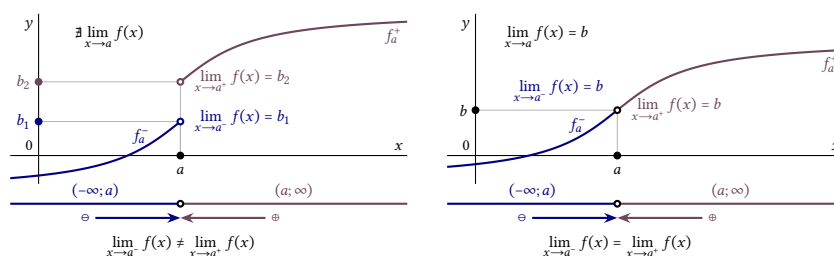
$$a \in \mathbb{R}, b \in \mathbb{R}^* \quad \bullet \quad \lim_{x \rightarrow a} f(x) = b \quad \Leftrightarrow \quad \bullet \quad \lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = b.$$

(%i3) limit(1/x,x,0,minus); limit(1/x,x,0); limit(1/x,x,0,plus);

(%o1) $-\infty$

(%o2) infinity /* Complex inf */

(%o3) ∞



One-sided limits (left) and two-sided limit (right).

Important limits.

$a, b \in \mathbb{R}, a > 0.$

$\bullet \quad \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$	$\bullet \quad \lim_{x \rightarrow \infty} \left(1 + \frac{b}{x}\right)^x = e^b.$	$\bullet \quad \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e.$
$\bullet \quad \lim_{x \rightarrow 0} \frac{\arcsin x}{x} = 1.$	$\bullet \quad \lim_{x \rightarrow 0} \sqrt[x]{1 + bx} = e^b.$	$\bullet \quad \lim_{x \rightarrow 0} \sqrt[x]{1 + x} = e.$
$\bullet \quad \lim_{x \rightarrow \infty} \sqrt[x]{x} = 1.$	$\bullet \quad \lim_{x \rightarrow \infty} x (\sqrt[x]{a} - 1) = \ln a.$	$\bullet \quad \lim_{x \rightarrow \infty} x (\sqrt[x]{e} - 1) = \ln e = 1.$
$\bullet \quad \lim_{x \rightarrow \infty} \sqrt[x]{a} = 1.$	$\bullet \quad \lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \ln a.$	$\bullet \quad \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = \ln e = 1.$

(%i2) limit(x*(%e^(1/x)-1),x,0); limit(x*(%e^(1/x)-1),x,inf);

(%o1) und /* undefined */

(%o2) 1

Example.

- $\bullet \quad \lim_{x \rightarrow 2} \frac{x-2}{x^2-3x+2} = \lim_{x \rightarrow 2} \frac{x-2}{(x-2)(x-1)} = \lim_{x \rightarrow 2} \frac{1}{x-1} = \frac{1}{2-1} = 1.$
- $\bullet \quad \lim_{x \rightarrow 0} \frac{3x+2x^{-1}}{x+4x^{-1}} = \lim_{x \rightarrow 0} \frac{3x+2x^{-1}}{x+4x^{-1}} \cdot \frac{x}{x} = \lim_{x \rightarrow 0} \frac{3x^2+2}{x^2+4} = \frac{3 \cdot 0 + 2}{0 + 4} = \frac{1}{2}.$
- $\bullet \quad \lim_{x \rightarrow 2} \frac{x^2-3x+2}{x^2-2x} = \lim_{x \rightarrow 2} \frac{(x-1)(x-2)}{x(x-2)} = \lim_{x \rightarrow 2} \frac{x-1}{x} = \frac{2-1}{2} = \frac{1}{2}. \blacksquare$

(%i3)	limit((x-2)/(x^2-3*x+2), x, 2);
	limit((3*x+2*1/x)/(x+4*1/x), x, 0);
	limit((x^2-3*x+2)/(x^2-2*x), x, 2);
(%o1)	1
(%o2)	$\frac{1}{2}$
(%o3)	$\frac{1}{2}$

Example.

- $$\lim_{x \rightarrow 0} \frac{x}{\sqrt{1+x}-\sqrt{1-x}} = \lim_{x \rightarrow 0} \frac{x \cdot (\sqrt{1+x} + \sqrt{1-x})}{(\sqrt{1+x}-\sqrt{1-x}) \cdot (\sqrt{1+x} + \sqrt{1-x})} = \lim_{x \rightarrow 0} \frac{x \cdot (\sqrt{1+x} + \sqrt{1-x})}{(1+x) - (1-x)}$$

$$= \lim_{x \rightarrow 0} \frac{x \cdot (\sqrt{1+x} + \sqrt{1-x})}{2x} = \lim_{x \rightarrow 0} \frac{\sqrt{1+x} + \sqrt{1-x}}{2} = \frac{\sqrt{1+0} + \sqrt{1-0}}{2} = \frac{1+1}{2} = 1.$$
- $$\lim_{x \rightarrow 0} \frac{1-\sqrt{1-x}}{x} = \lim_{x \rightarrow 0} \frac{(1-\sqrt{1-x}) \cdot (1+\sqrt{1-x})}{x \cdot (1+\sqrt{1-x})} = \lim_{x \rightarrow 0} \frac{1-(1-x)}{x \cdot \sqrt{x^2-x^3}} = \lim_{x \rightarrow 0} \frac{x}{x + \sqrt{x^2-x^3}}$$

$$= \lim_{x \rightarrow 0} \frac{x}{x + x \cdot \sqrt{1-x}} = \lim_{x \rightarrow 0} \frac{1}{1 + \sqrt{1-x}} = \frac{1}{1 + \sqrt{1-0}} = \frac{1}{2}.$$
- $$\lim_{x \rightarrow \infty} \frac{\sqrt{x^2-1} + \sqrt{x^2+1}}{x} = \lim_{x \rightarrow \infty} \left(\sqrt{\frac{x^2-1}{x^2}} + \sqrt{\frac{x^2+1}{x^2}} \right) = \lim_{x \rightarrow \infty} \left(\sqrt{1 - \frac{1}{x^2}} + \sqrt{1 + \frac{1}{x^2}} \right)$$

$$= \sqrt{1 - \frac{1}{\infty}} + \sqrt{1 + \frac{1}{\infty}} = \sqrt{1-0} + \sqrt{1+0} = 1 + 1 = 2. \blacksquare$$

(%i3)	limit(x/(sqrt(1+x)-sqrt(1-x)), x, 0);
	limit((1-sqrt(1-x))/x, x, 0);
	limit((sqrt(x^2-1)+sqrt(x^2+1))/x, x, inf);
(%o1)	1
(%o2)	$\frac{1}{2}$
(%o3)	2

Example.

- $$\lim_{x \rightarrow 1} \frac{\sqrt[3]{x}-1}{\sqrt[4]{x}-1} = \left[\text{Subst. } x = z^{12} \right] = \lim_{z \rightarrow 1} \frac{\sqrt[3]{z^{12}}-1}{\sqrt[4]{z^{12}}-1} = \lim_{z \rightarrow 1} \frac{z^4-1}{z^3-1}$$

$$= \lim_{z \rightarrow 1} \frac{(z-1)(z^3+z^2+z+1)}{(z-1)(z^2+z+1)} = \lim_{z \rightarrow 1} \frac{z^3+z^2+z+1}{z^2+z+1} = \frac{1+1+1+1}{1+1+1} = \frac{4}{3}.$$
- $$\lim_{x \rightarrow \infty} \left(\frac{5x^2}{x^2-1} + 2^{\frac{1}{x}} \right) = \lim_{x \rightarrow \infty} \frac{5}{1-x^{-2}} + \lim_{x \rightarrow \infty} 2^{\frac{1}{x}} = \frac{5}{1-\infty^{-2}} + 2^0 = \frac{5}{1-0} + 1 = 6.$$
- $$\lim_{x \rightarrow 0^+} x^{\frac{a}{\ln x}} = \lim_{x \rightarrow 0^+} e^{\ln x \cdot \frac{a}{\ln x}} = \lim_{x \rightarrow 0^+} e^{\frac{a}{\ln x} \cdot \ln x} = \lim_{x \rightarrow 0^+} e^a = e^a \text{ for } a \in \mathbb{R}. \blacksquare$$

(%i3)	limit((x^(1/3)-1)/(x^(1/4)-1), x, 1); limit(5*x^2/(x^2-1)+2^(1/x), x, inf); limit(x^(a/log(x)), x, 0, plus);
(%o1)	$\frac{4}{3}$
(%o2)	6
(%o3)	e^a

If we use the substitution $x = z^{12}$, we can simplify the first limit.

(%i2)	f(x):=(x^(1/3)-1)/(x^(1/4)-1)\$ g(z):=subst(z^12,x,f(x))\$ 'limit(g(z),z,1); limit(g(z),z,1);
(%o1)	$\lim_{z \rightarrow 1} \frac{z^4-1}{z^2 z -1}$ /* z is positive, z= z */
(%o2)	$\frac{4}{3}$

In the last example, we calculated the limit of the expression 0^0 – the so-called indefinite expression.

Among the **indeterminate expressions** (we count them using limits) are:

$$\bullet \infty - \infty, \bullet \pm\infty \cdot 0, \bullet \frac{0}{0}, \bullet \frac{1}{0}, \bullet \frac{\pm\infty}{0}, \bullet \frac{\pm\infty}{\pm\infty}, \bullet 0^0, \bullet 0^{\pm\infty}, \bullet 1^{\pm\infty}, \bullet (\pm\infty)^0.$$

Example.

$$\bullet \lim_{x \rightarrow \infty} x(\ln(x+2) - \ln x) = \lim_{x \rightarrow \infty} x \cdot \ln \frac{x+2}{x} = \lim_{x \rightarrow \infty} \ln \left(1 + \frac{2}{x}\right)^x = \ln e^2 = 2.$$

$$\bullet \lim_{x \rightarrow 0} \frac{x}{\ln(1+tx)} = \left[\begin{array}{l} \text{Subst. } z = tx \\ x \rightarrow 0, z \rightarrow 0 \end{array} \right] = \lim_{z \rightarrow 0} \frac{\frac{z}{t}}{\ln(1+z)} = \frac{1}{t} \cdot \lim_{z \rightarrow 0} \frac{1}{\frac{1}{z} \cdot \ln(1+z)}$$

$$= \frac{1}{t} \cdot \lim_{z \rightarrow 0} \frac{1}{\ln(1+z)^{\frac{1}{z}}} = \frac{1}{t} \cdot \frac{1}{\ln e} = \frac{1}{t} \cdot \frac{1}{1} = \frac{1}{t} \text{ for } t \in \mathbb{R}, t \neq 0.$$

$$\bullet \lim_{x \rightarrow \infty} \left(\frac{3x-2}{3x+1}\right)^x = \lim_{x \rightarrow \infty} \left(\frac{3x+1-3}{3x+1}\right)^{\frac{3x+1-1}{3}} = \left[\begin{array}{l} \text{Subst. } z = 3x+1 \\ x \rightarrow \infty, z \rightarrow \infty \end{array} \right] = \lim_{z \rightarrow \infty} \left(\frac{z-3}{z}\right)^{\frac{z-1}{3}}$$

$$= \lim_{z \rightarrow \infty} \left[\left(1 - \frac{3}{z}\right)^z\right]^{\frac{z-1}{3z}} = [e^{-3}]^{\frac{1}{3}} = e^{-1} = \frac{1}{e}. \blacksquare$$

(%i3)	limit(x*(log(x+2)-log(x)), x, inf); limit(x/log(1+t*x), x, 0); limit(((3*x-2)/(3*x+1))^x, x, inf);
(%o1)	2
(%o2)	$\frac{1}{t}$
(%o3)	e^{-1}

Asymptotic Properties

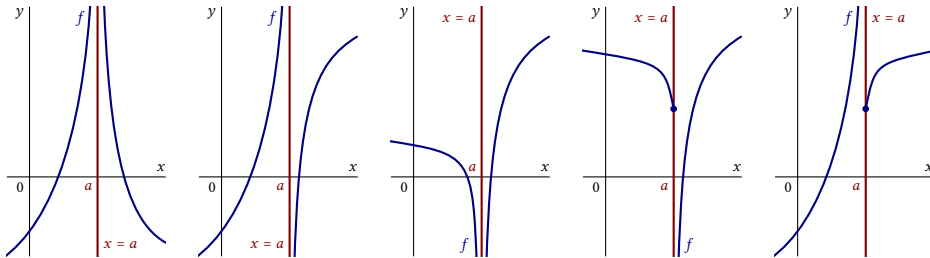
When investigating the function f , it is important to examine its properties at non-eigenpoints:

- For $x \rightarrow \pm\infty$.
- In the neighborhood $O(a)$ of the point $a \in R$, for which holds $\lim_{x \rightarrow a^-} f(x) = \pm\infty$ or $\lim_{x \rightarrow a^+} f(x) = \pm\infty$.

$$y = f(x), x \in D(f), a \in R.$$

- The line $x = a$ is called **asymptote without slope (vertical)** of the graph f ,
if $\lim_{x \rightarrow a^-} f(x) = \pm\infty$ or $\lim_{x \rightarrow a^+} f(x) = \pm\infty$ (at least one of the limits is infinite).
- The line $y = kx + q$ is called the **asymptote with slope** of the graph f ,
if $\lim_{x \rightarrow -\infty} [f(x) - (kx + q)] = 0$ or $\lim_{x \rightarrow \infty} [f(x) - (kx + q)] = 0$.

If $k = 0$, i.e. $\lim_{x \rightarrow -\infty} f(x) = q$ or $\lim_{x \rightarrow \infty} f(x) = q$,
the asymptote $y = q$ is called **horizontal (horizontal)**.



Examples of asymptotes without slope

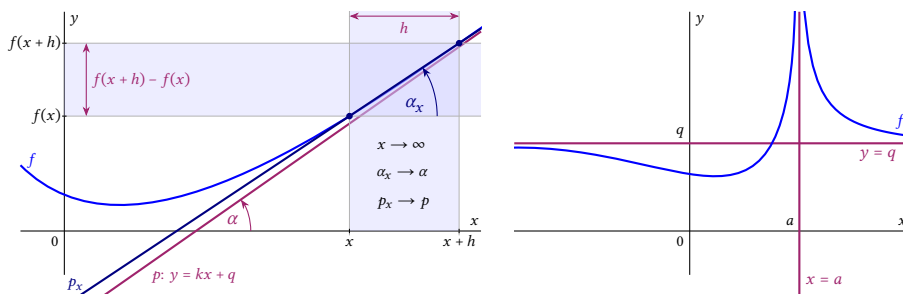
The functions $y = f(x)$, $x \in D(f)$ and $D(f)$ is an unbounded set.

- The line $y = kx + q$ is an asymptote with the slope of the graph f .

$$\Leftrightarrow \bullet \text{ Exist } \lim_{x \rightarrow \pm\infty} \frac{f(x)}{x} = k, \lim_{x \rightarrow \pm\infty} [f(x) - kx] = q, k, q \in R.$$

- $\lim_{x \rightarrow \infty} \frac{f(x) - (kx + q)}{x} = \lim_{x \rightarrow \infty} \left[\frac{f(x)}{x} - k - \frac{q}{x} \right] = 0. \quad \Rightarrow \bullet \lim_{x \rightarrow \infty} \frac{f(x)}{x} = k.$
- $\lim_{x \rightarrow \infty} [f(x) - (kx + q)] = \lim_{x \rightarrow \infty} [(f(x) - kx) - q] = 0. \quad \Rightarrow \bullet \lim_{x \rightarrow \infty} [f(x) - kx] = q.$

```
(%i10) f(x):=(2*x^2+x+1)/(8*x); km:limit(f(x)/x,x,minf)$
kp:limit(f(x)-kx,x,inf)$
```



Asymptote with slope α (left), vertical $y = q$ and horizontal asymptote $x = a$ (right)

```

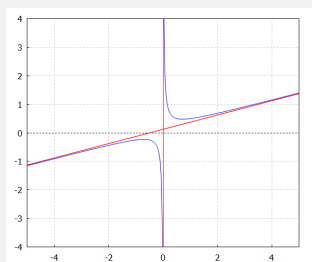
qm: limit(f(x)-km*x, x, minf) $ qp: limit(f(x)-kp*x, x, inf) $
dm(x) := km*x+qm $ dp(x) := kp*x+qp $ dm(x); dp(x);
draw2d(grid=true, xaxis=true, yaxis=true, xrange=[-5, 5], yrange=[-4, 4],
color=blue, explicit(f(x), x, -8, 0), explicit(f(x), x, 0, 8),
color=red, parametric(0, t, t, -5, 5),
explicit(dm(x), x, -8, 8), explicit(dp(x), x, -8, 8)) $

```

(%01) $f(x) := \frac{2x^2+x+1}{8x}$

(%08) $\frac{x}{4} + \frac{1}{8}$

(%09) $\frac{x}{4} + \frac{1}{8}$



Continuity of function

The concept of the limit of the function f at the point a is closely related to the concept of the continuity of f at the point a . Continuity is also a local matter in some neighborhood of $O(a)$.

The function f is continuous at the point $a \in D(f)$ if:

- For all $\{x_n\}_{n=1}^{\infty} \subset D(f)$, $\{x_n\}_{n=1}^{\infty} \rightarrow a$ holds $\{f(x_n)\}_{n=1}^{\infty} \rightarrow f(a)$.

$$y = f(x), x \in D(f).$$

- $a \in D(f)$ is an isolated point. \implies • The function f is continuous at point a .

When $a \in D(f)$ is an isolated point, then there exists a single $\{x_n\}_{n=1}^{\infty} = \{a\}_{n=1}^{\infty} \longrightarrow a$. The definition of continuity at the point $a \in D(f)$ using sequences is called Heine. We can characterize the continuity using the neighborhood $O(a)$ and $O(f(a))$.

The function f is **continuous at the point** $a \in D(f)$ if:

- For every neighborhood $O(f(a))$ there is a neighborhood $O(a)$ such that for all $x \in O(a)$ holds $f(x) \in O(f(a))$, i.e. $f(O(a)) \subset O(f(a))$.

If $a \in D(f)$ is an accumulation point, then the definition of continuity coincides with the definition of limit.

$$y = f(x), x \in D(f), a \in D(f) \text{ is an accumulation point } D(f).$$

- The function f is continuous at point a . \iff • $\lim_{x \rightarrow a} f(x) = f(a)$.

The function f can be discontinuous only at the accumulation point $a \in R$ (point of discontinuity), while $a \in D(f)$ may not hold.

Removable discontinuity

$$\text{There exists } \lim_{x \rightarrow a} f(x) = b \in R, b \neq f(a).$$

Non-removable discontinuity of the type I.

$$\left. \begin{array}{l} \text{There exist } \lim_{x \rightarrow a^-} f(x) = b^- \in R \\ \text{and } \lim_{x \rightarrow a^+} f(x) = b^+ \in R \end{array} \right\} b^- \neq b^+.$$

The difference $c = b^+ - b^-$ is called the jump of the function f at the point a .

Non-removable discontinuity of the type II.

$$\left. \begin{array}{l} \text{At least one of } \lim_{x \rightarrow a^-} f(x) \\ \text{or } \lim_{x \rightarrow a^+} f(x) \end{array} \right\} \text{ does not exist or is infinite.}$$

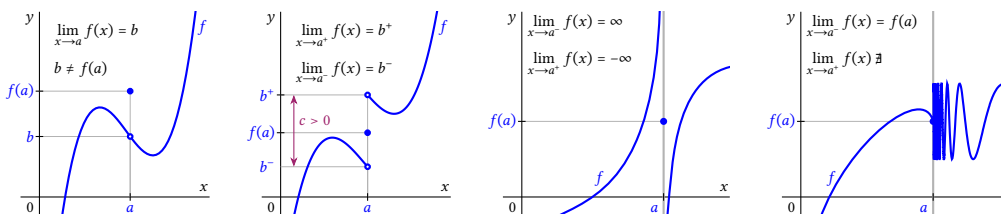
Asymptotic discontinuity, if one of the one-sided limits is infinite.

The function f is discontinuous at the point $a \in R$.

Value $f(a)$ may, but may not exist.

The functions f, g are continuous at the point $a \in D(f) \cap D(g), r \in R$.

$$\implies |f|, f \pm g, rf, fg \text{ and } \frac{f}{g} \text{ for } g(a) \neq 0 \text{ are continuous at point } a.$$



Discontinuity of function f at point a : removable, non-removable type I, non-removable type II, non-removable type II (from left to right).

Continuity of a composite function.

A function f is continuous at the point $a \in D(f)$.
 A function g is continuous at the point $b = f(a) \in D(g)$.
 $H(f) \subset D(g)$.
 $\left. \begin{array}{l} \text{A function } f \text{ is continuous at the point } a \in D(f). \\ \text{A function } g \text{ is continuous at the point } b = f(a) \in D(g). \\ H(f) \subset D(g). \end{array} \right\} \Rightarrow \text{The function } F = g(f) \text{ is continuous at point } a.$

$y = f(x)$, $x \in D(f)$, point $a \in D(f)$.

$$f_a^-(x) = f(x)|_{D(f) \cap (-\infty; a)} = f(x)|_{\{x \in D(f), x \leq a\}}$$

Narrowing the function f to the left.

$$f_a^+(x) = f(x)|_{D(f) \cap (a; \infty)} = f(x)|_{\{x \in D(f), a \leq x\}}$$

Narrowing the function f to the right.

$f_a^-(x)$ continuous at point a Continuity from left.
 $f_a^+(x)$ continuous at point a Continuity from right.
 $f(x)$ continuous at point a Two-side continuity of the function f at the point a .

} One-sided continuity
of the function f at the point a .

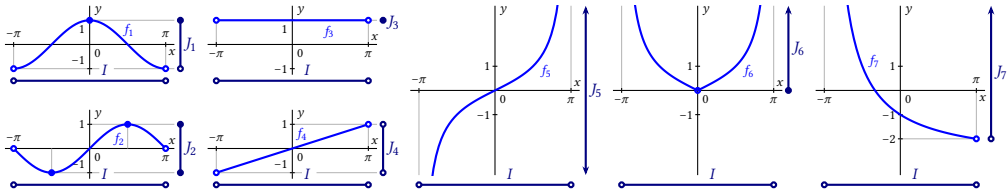
- A function f is continuous at the point $a \in D(f)$.
 \Rightarrow • There exists $O(a)$ in which f is bounded.

A function f is called **continuous on the set** $A \subset D(f)$, if it is continuous at every point $a \in A$.

- A function f is continuous on the set $A \subset D(f)$.
 \Rightarrow • The function f need not be bounded on A .

- A function f is continuous on the interval $I \subset R$. \Rightarrow • The set $f(I)$ is an interval.

- A interval $I = \langle a; b \rangle$ is closed. \Rightarrow • The set $f(I)$ is a closed interval.
- A interval I is not closed. \Rightarrow • The set $f(I)$
can be an interval of different types.



Examples of mapping the interval $I = (-\pi; \pi)$ by continuous functions, $f_1(x) = \cos x: I \rightarrow J_1 = (-1; 1)$,
 $f_2(x) = \sin x: I \rightarrow J_2 = (-1; 1)$, $f_3(x) = 1: I \rightarrow J_3 = \{1\}$, $f_4(x) = \frac{x}{\pi}: I \rightarrow J_4 = (-1; 1)$,
 $f_5(x) = \tan \frac{x}{2}: I \rightarrow J_5 = (-\infty; \infty)$, $f_6(x) = \left| \tan \frac{x}{2} \right|: I \rightarrow J_6 = (0; \infty)$, $f_7(x) = -\frac{3x+\pi}{x+\pi}: I \rightarrow J_7 = (-2; \infty)$.

A function f is called **piecewise continuous on the set** $A \subset D(f)$, if it is continuous on the set A , except for a finite number of points of discontinuity (removable or non-removable type I).

- $f: y = \frac{1}{x}, x \in \mathbb{R}$ is continuous on the interval $(0; 1)$, but it is not bounded to $(0; 1)$.

Cauchy's zero point theorem.

A function f is continuous on $\langle a; b \rangle$.
 $f(a) \cdot f(b) < 0$. } \Rightarrow • There exists $c \in (a; b)$ such that $f(c) = 0$.

Chapter 3

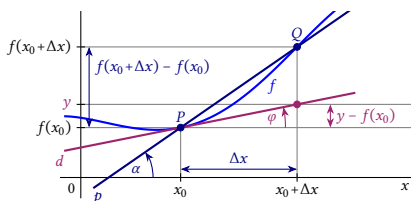
Differential Calculus

Derivative of a real function

Example.

The function $y = f(x)$, $x \in D(f)$ is continuous.

- The points $P = [x_0; f(x_0)]$, $Q = [x_0 + \Delta x; f(x_0 + \Delta x)]$ lie on graph f .
- The line PQ has the slope $\tan \alpha = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$.
- The tangent line to f at point P has the form $d_P: y - f(x_0) = \tan \varphi \cdot \Delta x$,
where $\tan \varphi = \frac{y - f(x_0)}{\Delta x}$ is its slope.
- $Q \rightarrow P. \Rightarrow PQ \rightarrow d_P, \Delta x \rightarrow 0, \alpha \rightarrow \varphi, f(x_0 + \Delta x) \rightarrow f(x_0). \Rightarrow \tan \alpha \rightarrow \tan \varphi.$
 \Rightarrow • The tangent line $\tan \varphi = \lim_{\alpha \rightarrow \varphi} \tan \alpha = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$. ■



- $Q \rightarrow P.$
 \Rightarrow • $PQ \rightarrow d.$ • $\alpha \rightarrow \varphi.$ • $\tan(\alpha) \rightarrow \tan(\varphi).$
 \Rightarrow • $\tan \varphi = \lim_{\alpha \rightarrow \varphi} \tan \alpha = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$.

Geometric meaning of the derivative of a function at a point.

$y = f(x)$, $x \in D(f)$ has a **derivative at the point** $x_0 \in D(f)$, label $f'(x_0)$, or $y'(x_0)$,

$$\text{if it exists } \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \left[\begin{array}{l} \text{Subst. } h = x - x_0 \\ x \rightarrow x_0, \quad h \rightarrow 0 \end{array} \right] = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} = f'(x_0).$$

$$f'(x_0) = \frac{df(x_0)}{dx}, \text{ respectively } y'(x_0) = \frac{dy(x_0)}{dx} \text{ (label using differentials).}$$

$f'(x_0) \in R$. Eigen (finite) } derivative of the function f
 $f'(x_0) = \infty$ or $f'(x_0) = -\infty$. Non-eigen (infinite) } at the point x_0 .

Function $y = f(x)$, $x \in D(f)$, point $x_0 \in D(f)$.

- There exists $f'(x_0) \in R$ (finite). \implies • f is continuous at the point x_0 .

The continuity of the function f at the point x_0 does not guarantee the existence of $f'(x_0)$.

Example.

The function $f: y = |x|$ is continuous at the point $x_0 = 0$.

- There does not exist

$$f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{|x| - |0|}{x - 0} = \lim_{x \rightarrow 0} \frac{|x|}{x} = \begin{cases} \lim_{x \rightarrow 0^+} \frac{x}{x} = 1. \\ \lim_{x \rightarrow 0^+} \frac{-x}{x} = -1. \blacksquare \end{cases}$$

$f'(x_0)$ represents geometrically the tangent line to the graph f at the point x_0 .

$f'(x_0) \in R$. Tangent line $d: y = f(x_0) + f'(x_0)(x - x_0)$
 with slope $f'(x_0)$.

$f'(x_0) = \pm\infty$.
 f is continuous at the point x_0 . } Tangent line $d: x = x_0$ without slope (vertical).

We calculate and simplify the derivative of the function $f(x) = \ln(x + \sqrt{x^2 + 1})$.

```
(%i1) f(x) := log(x+sqrt(x^2+1));
(%o1) f(x) := log(x + sqrt(x^2 + 1))
(%i3) f_1(x) := diff(f(x), x); f_1(x);
(%o2) f_1(x) := d/dx f(x)
(%o3) (sqrt(x^2+1)+1)/sqrt(x^2+1+x)
(%i4) ratsimp(f1(x));
(%o4) (sqrt(x^2+1)+x)/(x*sqrt(x^2+1)+x^2+1)
```

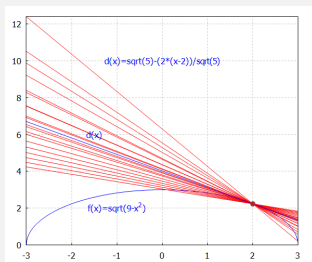
We have calculated the derivative of $f'(x)$, but we have not succeeded in simplifying it appropriately. We will use the command `subst`.

```
(%i5) fp: subst(a, sqrt(x^2+1), f1(x));
(fp) (x+1)/(x+a)
(%i6) ratsimp(fp);
```

```
(%O6)  $\frac{1}{a}$ 
(%i7) subst(sqrt(x^2+1), a, ratsimp(fp));
(%O7)  $\frac{1}{\sqrt{x^2+1}}$ 
```

We determine the tangent line to the semicircle $y = \sqrt{9 - x^2}$ at the point 2.

```
(%i8) f(x):=sqrt(9-x^2)$ pmer(a,b):=(f(b)-f(a))/(b-a)$
sek(x,a,b):=pmer(a,b)*(x-a)+f(a)$
Sek:=makelist(explicit(sek(x,2,-.15+.25*i),x,-3,3),i,1,20)$
f1(x):=diff(f(x),x,1)$ d(x):=f(2)+subst(2,x,f1(x))*(x-2)$
print("Secant y=d(x)=",d(x)," in point 2 have a blue color")$
draw2d(grid=true,xaxis=true,color=blue,explicit(f(x),x,-3,3),
color=red,Sek,color=blue,explicit(d(x),x,-3,3),
point_type=7,color=brown,points([[2,f(2)]]),
color=blue,label(["f(x)=sqrt(9-x^2)",-1,2]),
label(["d(x)",-1.5,6]),
label([concat("d(x)=",string(d(x))),0,10]))$
Secant  $y = d(x) = \sqrt{5} - \frac{2(x-2)}{\sqrt{5}}$  in point 2 have a blue color
```

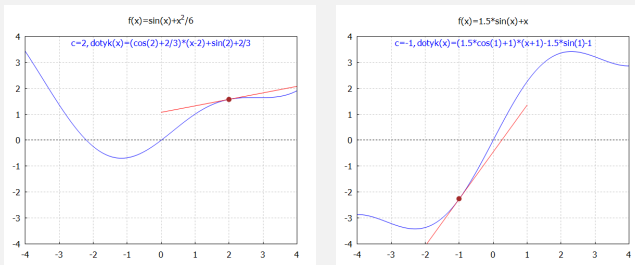


$y = f(x)$, $x \in D(f)$, point $x_0 \in D(f)$.

$$\begin{array}{l}
 f'_-(x_0) = \lim_{x \rightarrow x_0^-} \frac{f(x) - f(x_0)}{x - x_0} \quad \text{Derivative from left.} \\
 f'_+(x_0) = \lim_{x \rightarrow x_0^+} \frac{f(x) - f(x_0)}{x - x_0} \quad \text{Derivative from right.} \\
 f'(x_0) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \quad \text{Two-sided derivative of the function } f \text{ at the point } x_0.
 \end{array}
 \left. \begin{array}{l} \\ \\ \end{array} \right\} \begin{array}{l} \text{One-sided derivatives} \\ \text{of the function } f \text{ at the point } x_0. \end{array}$$

The following construction calculates and draws the tangent line to the graph of the function f at point c .

```
(%i6) c:2$ f(x):=x^2/6+sin(x)$
f1(x):=diff(f(x),x,1)$ d(x):=f(c)+subst(c,x,f1(x))*(x-c)$
print("Secant y=d(x)=",d(x)," in point",c)$
draw2d(grid=true,xaxis=true,xrange=[-4,4],yrange=[-4,4],
color=blue,explicit(f(x),x,-4,4),
color=red,explicit(d(x),x,c-2,c+2),
point_type=7,color=brown,points([[c,f(c)]]),
color=blue,title=concat("f(x)=",string(f(x))),
label([concat("c=",string(c)," ",d(x)=",",string(d(x))),0,3.75]))$
  Secant y = d(x) = (cos(2) + 2/3) * (x - 2) + sin(2) + 2/3 in point 2
  Secant y = d(x) = (cos(2) + 2/3)(x - 2) + sin(2) + 2/3 in point 2
```



Function $y = f(x)$, $x \in D(f)$.

- $x_0 \in D(f)$. \Rightarrow • $f'(x_0)$ is a number or $\pm\infty$.
- $A \subset \{x_0 \in D(f), f'(x_0) \text{ is finite}\}, A \neq \emptyset$. \Rightarrow • $y = f'(x)$, $x \in A$ is a function.

It is called the **derivative** of the function f on the set A , label $f' = \frac{df}{dx}$, respectively $y' = \frac{dy}{dx}$.

Function $y = f(x)$, $x \in D(f)$, set $A \subset D(f)$.

- $\forall x_0 \in A: f'(x_0) \in \mathbb{R}$ (finite derivative). \Rightarrow • f is continuous on the set A .

Example.

- $f : y = x^n$, $x \in \mathbb{R}$, $n \in \mathbb{N}$, $x_0 \in D(f)$.

$$\begin{aligned} f'(x_0) &= \lim_{x \rightarrow x_0} \frac{x^n - x_0^n}{x - x_0} = \lim_{x \rightarrow x_0} \frac{(x^{n-1} + x^{n-2}x_0 + \dots + x_0^{n-1})(x - x_0)}{x - x_0} \\ &= \lim_{x \rightarrow x_0} (x^{n-1} + x^{n-2}x_0 + \dots + x_0^{n-1}) = x_0^{n-1} + x_0^{n-2}x_0 + \dots + x_0^{n-1} = nx_0^{n-1}. \end{aligned}$$

- $f : y = e^x$, $x \in \mathbb{R}$.

$$[e^x]' = \lim_{h \rightarrow 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \rightarrow 0} \frac{e^x \cdot (e^h - 1)}{h} = e^x \cdot \lim_{h \rightarrow 0} \frac{e^h - 1}{h} = e^x \cdot 1 = e^x. \blacksquare$$

In the practical calculation of derivatives, we use various formulas and rules.

Rules for derivation.

Functions f, g have derivatives f', g' on $A \neq \emptyset$, point $x_0 \in A$, number $c \in \mathbb{R}$. \Rightarrow

- $(cf)'(x_0) = cf'(x_0)$.
- $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$.
- $(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)$.
- $\left[\frac{f}{g}\right]'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g^2(x_0)}$ for $g(x_0) \neq 0$.

$$\bullet (cf)' = cf' \quad \bullet (f \pm g)' = f' \pm g' \quad \bullet (fg)' = f'g + fg' \quad \bullet \left[\frac{f}{g}\right]' = \frac{f'g - fg'}{g^2}$$

Example.

$f: y = \frac{x}{x-1}, x \in \mathbb{R} - \{1\}$, straight line $p: y = 2 - x$.

The tangent lines to the graph f parallel to p are $d_1: y = -x$, $d_2: y = 4 - x$.

- The tangent $d: y = f(x_0) + f'(x_0) \cdot (x - x_0)$ at the point x_0 has the slope $f'(x_0)$.
- The line p has the slope -1 . $\Rightarrow f'(x_0) = -1$.
- $f'(x) = \left(\frac{x}{x-1}\right)' = \frac{1 \cdot (x-1) - x(1-0)}{(x-1)^2} = \frac{x-1-x}{(x-1)^2} = \frac{-1}{(x-1)^2}, x \in \mathbb{R} - \{1\}$.
- $f'(x_0) = \frac{-1}{(x_0-1)^2} = -1$. $\Rightarrow (x_0 - 1)^2 = 1$. $\Rightarrow x_0 = 0$ or $x_0 = 2$.
- Two tangent points $D = [x_0; f(x_0)]$ and two tangent lines d :

$$D_1 = [0; 0], d_1: y = 0 - (x - 0) = -x.$$

$$D_2 = [2; 2], d_2: y = 2 - (x - 2) = 4 - x. \blacksquare$$

The derivative of the inverse function.

$y = f(x), x \in I$ is a bijection, $I \subset \mathbb{R}$ is an interval, $x_0 \in I$ is an interior point.

$$\left. \begin{array}{l} f \text{ is continuous on } I. \\ f'(x_0) \neq 0 \text{ is finite.} \end{array} \right\} \Rightarrow \bullet \text{ Exists } [f^{-1}]'(y_0) = \frac{1}{f'(x_0)} \Big|_{x_0=f^{-1}(y_0)} = \frac{1}{f'(f^{-1}(y_0))}.$$

$$[f^{-1}]'(y) = \frac{1}{f'(x)} \Big|_{x=f^{-1}(y)} = \frac{1}{f'(f^{-1}(y))}, \quad \frac{df^{-1}(y)}{dy} = \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{\frac{df(x)}{dx}}$$

Example.

$f: y = e^x, x \in \mathbb{R}$ is continuous and increasing, $f'(x) = e^x \neq 0$ for $x \in \mathbb{R}$.

$f^{-1}: x = \ln y$ for $y \in (0; \infty)$.

$$\bullet [\ln y]' = [f^{-1}]'(y) = \frac{1}{f'(x)} = \frac{1}{[e^x]'} = \frac{1}{e^x} = \frac{1}{e^{\ln y}} = \frac{1}{y} \text{ for } y \in (0; \infty). \blacksquare$$

Example.

$f: y = \sin x, x \in \left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$ is continuous and increasing, $H(f) = (-1; 1)$.

$$\bullet f'(x) = \cos x = \sqrt{1 - \sin^2 x} > 0 \text{ for } x \in \left(-\frac{\pi}{2}; \frac{\pi}{2}\right).$$

$$\bullet [\arcsin y]' = \frac{1}{[\sin x]'} = \frac{1}{\cos x} = \frac{1}{\sqrt{1 - \sin^2 x}} = \frac{1}{\sqrt{1 - [\sin \arcsin y]^2}} = \frac{1}{\sqrt{1 - y^2}}, y \in (-1; 1). \blacksquare$$

The derivative of a composite function.

$u = f(x), x \in D(f), y = g(u), u \in D(g), H(f) \subset D(g), y = F(x) = g(f(x)), x \in D(f)$.

$$\left. \begin{array}{l} x_0 \in D(f), u_0 = f(x_0). \\ f'(x_0), g'(u_0) \text{ are finite.} \end{array} \right\} \Rightarrow \bullet F'(x_0) = [g(f(x_0))]' = g'(f(x_0)) \cdot f'(x_0).$$

$$F'(x) = [g(f)]'(x) = g'(u) \cdot f'(x), \quad \frac{dF(x)}{dx} = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{dg(u)}{du} \cdot \frac{df(x)}{dx}.$$

Example.

$$\bullet [\sin(\sin x)]' = \cos(\sin x) \cdot [\sin x]' = \cos(\sin x) \cdot \cos x, x \in \mathbb{R}.$$

$$\bullet [\sin(\sin(\sin x))]' = \cos(\sin(\sin x)) \cdot [\sin(\sin x)]' \\ = \cos(\sin(\sin x)) \cdot \cos(\sin x) \cdot [\sin x]' = \cos(\sin(\sin x)) \cdot \cos(\sin x) \cdot \cos x, x \in \mathbb{R}.$$

$$\bullet [a^x]' = [e^{\ln a^x}]' = [e^{x \ln a}]' = e^{x \ln a} \cdot [x \ln a]' = a^x \cdot \ln a, x \in \mathbb{R}, a > 0, a \neq 1.$$

$$\bullet [x^a]' = [e^{\ln x^a}]' = [e^{a \ln x}]' = e^{a \ln x} \cdot [a \ln x]' = x^a \cdot \frac{a}{x} = ax^{a-1}, x > 0, a \in \mathbb{R}.$$

$$\bullet [x^x]' = [e^{\ln x^x}]' = [e^{x \ln x}]' = e^{x \ln x} \cdot [x \ln x]' \\ = x^x \cdot \left[1 \cdot \ln x + x \cdot \frac{1}{x}\right] = x^x \cdot [1 + \ln x], x > 0. \blacksquare$$

Logarithmic derivative of the function f .

$y = f(x), x \in D(f)$, point $x_0 \in D(f)$.

$$\left. \begin{array}{l} f(x_0) > 0. \\ \text{There exists } f'(x_0). \end{array} \right\} \Rightarrow \bullet [\ln f(x_0)]' = \frac{f'(x_0)}{f(x_0)}. \bullet f'(x_0) = f(x_0) \cdot [\ln f(x_0)]'.$$

Derivatives of basic elementary functions

$[c]' = 0$	$x \in \mathbb{R}, c \in \mathbb{R}$	$[x]' = 1$	$x \in \mathbb{R}$
$[x^n]' = nx^{n-1}$	$x \in \mathbb{R}, n \in \mathbb{N}$	$[x^a]' = ax^{a-1}$	$x > 0, a \in \mathbb{R}$
$[e^x]' = e^x$	$x \in \mathbb{R}$	$[a^x]' = a^x \ln a$	$x \in \mathbb{R}, a > 0$
$[\ln x]' = \frac{1}{x}$	$x > 0$	$[\log_a x]' = \frac{1}{x \ln a}$	$x > 0, a > 0, a \neq 1$
$[\sin x]' = \cos x$	$x \in \mathbb{R}$	$[\cos x]' = -\sin x$	$x \in \mathbb{R}$
$[\tan x]' = \frac{1}{\cos^2 x}$	$x \neq \frac{(2k+1)\pi}{2}, k \in \mathbb{Z}$	$[\cot x]' = -\frac{1}{\sin^2 x}$	$x \neq k\pi, k \in \mathbb{Z}$
$[\arcsin x]' = \frac{1}{\sqrt{1-x^2}}$	$x \in (-1; 1)$	$[\arccos x]' = -\frac{1}{\sqrt{1-x^2}}$	$x \in (-1; 1)$
$[\arctan x]' = \frac{1}{1+x^2}$	$x \in \mathbb{R}$	$[\operatorname{arccot} x]' = -\frac{1}{1+x^2}$	$x \in \mathbb{R}$
$[\sinh x]' = \cosh x$	$x \in \mathbb{R}$	$[\cosh x]' = \sinh x$	$x \in \mathbb{R}$
$[\tanh x]' = \frac{1}{\cosh^2 x}$	$x \in \mathbb{R}$	$[\operatorname{coth} x]' = -\frac{1}{\sinh^2 x}$	$x \neq 0$
$[\operatorname{arsinh} x]' = \frac{1}{\sqrt{x^2+1}}$	$x \in \mathbb{R}$	$[\operatorname{arcosh} x]' = \frac{1}{\sqrt{x^2-1}}$	$x > 1$
$[\operatorname{artanh} x]' = \frac{1}{1-x^2}$	$x \in (-1; 1)$	$[\operatorname{arcoth} x]' = \frac{1}{1-x^2}$	$x \in \mathbb{R} - \langle -1; 1 \rangle$

Derivatives of elementary functions are the basis of successful differentiation. For practical needs, it is necessary to remember the formulas from the table.

Differential of a function and derivatives of higher orders

We often need to approximate (approximately express) the given function f by another, simpler function g so that their difference $|f(x) - g(x)|$ was as small as possible. Most of the time, **local approximation** is enough in some neighborhood $O(x_0)$ of the point $x_0 \in D(f)$.

$y = f(x)$, $x \in D(f)$, point $x_0 \in D(f)$, there exists a finite $f'(x_0)$.

$$\left. \begin{aligned} df(x_0, x - x_0) &= f'(x_0) \cdot (x - x_0), x \in \mathbb{R}. \\ df(x_0, h) &= f'(x_0) \cdot h, h \in \mathbb{R}. \end{aligned} \right\} \begin{array}{l} \text{Differential of the function } f \\ \text{at the point } x_0, \\ f \text{ is differentiable at the point } x_0. \end{array}$$

$f: y = x$, $x \in \mathbb{R}$, point $x_0 \in \mathbb{R}$, $f'(x_0) = 1$ (finite).

$$\Rightarrow \bullet dx = df(x_0, h) = f'(x_0) \cdot h = 1 \cdot h = h, h \in \mathbb{R}.$$

$y = f(x)$, $x \in \mathbb{R}$, point $x_0 \in \mathbb{R}$, $f'(x_0)$ is finite.

$$\Rightarrow \bullet df(x_0) = df(x_0, h) = f'(x_0) \cdot h = f'(x_0) \cdot dx, h \in \mathbb{R}.$$

$$df(x_0) = f'(x_0) \cdot dx. \Rightarrow f'(x_0) = \frac{df(x_0)}{dx}, \text{ respectively } f' = \frac{df}{dx}.$$

On the best local linear approximation.

$y = f(x)$, $x \in D(f)$ is differentiable at the point $x_0 \in D(f)$.

- Approximation of the function f in the neighborhood $O(x_0)$ using at the point x_0 the tangent line $d: y = f(x_0) + f'(x_0)(x - x_0) = f(x_0)$, $x \in O(x_0)$ is the best of all approximations using a linear function (straight line).

Example.

$\sqrt[6]{1.06} \approx 1.01$.

Exactly $\sqrt[6]{1.06} = 1.0097588$, calculation error < 0.00025 .

- Let us denote $f(x) = \sqrt[6]{x}$, $x > 0$, $x_0 = 1$.

$$\Rightarrow \bullet f'(x) = [x^{1/6}]' = \frac{1}{6}x^{-5/6} = \frac{1}{6\sqrt[6]{x^5}}, x > 0, f'(x_0) = f'(1) = \frac{1}{6}.$$

- Let $O(1)$ be such that $1.06 \in O(1)$.

$$\Rightarrow \bullet \sqrt[6]{x} = f(x) \approx f(1) + f'(1) \cdot (x - 1) = 1 + \frac{x-1}{6} = \frac{6+x-1}{6} = \frac{x+5}{6}.$$

$$\Rightarrow \bullet \sqrt[6]{1.06} = f(1.06) \approx \frac{1.06+5}{6} = \frac{6.06}{6} = 1.01. \blacksquare$$

```
(%i8) c:1.06$ f(x):=x^(1/6)$
s:1$ f1(x):=diff(f(x),x,1)$
p(x):=f(s)+subst(s,x,f1(x))*(x-s)$ p(x);
h(c):=print("c=",c,"
c^(1/6)=","f(c),"=",float(f(c)),"approx",
subst(c,x,float(p(x))))$ h(c)$
(%o6)  $\frac{x-1}{6} + 1$ 
c = 1.06 c^(1/6) = f(1.06) = 1.009758794179192 approx 1.01
```

The approximation of the function f makes sense only for x near the point x_0 .

```
(%i18) h(0.9)$ h(1.1)$ h(1.2)$ h(1.5)$ h(2.0)$
h(4.0)$ h(10)$ h(16)$ h(32)$ h(64)$
c = 0.9 c^(1/6) = f(0.9) = 0.9825931938526898 approx 0.9833333333333334
c = 1.1 c^(1/6) = f(1.1) = 1.016011867773387 approx 1.0166666666666667
c = 1.2 c^(1/6) = f(1.2) = 1.030853320886445 approx 1.0333333333333333
c = 1.5 c^(1/6) = f(1.5) = 1.069913193933663 approx 1.0833333333333333
c = 2.0 c^(1/6) = f(2.0) = 1.122462048309373 approx 1.1666666666666667
c = 4.0 c^(1/6) = f(4.0) = 1.259921049894873 approx 1.5
```


$$\begin{aligned}
c = 10 & \quad c^{(1/6)} = f(10) = 1.46779926762207 \text{ approx } 2.5 \\
c = 16 & \quad c^{(1/6)} = f(16) = 1.587401051968199 \text{ approx } 3.5 \\
c = 32 & \quad c^{(1/6)} = f(32) = 1.781797436280679 \text{ approx } 6.166666666666666 \\
c = 64 & \quad c^{(1/6)} = f(64) = 2.0 \text{ approx } 11.5
\end{aligned}$$

Example (Other solution).

$$\sqrt[6]{1,06} \approx 1,01.$$

$$\text{Exactly } \sqrt[6]{1,06} = 1.0097588, \text{ calculation error } < 0.00025.$$

- Let us denote $f(x) = \sqrt[6]{x+1}$, $x > -1$, $x_0 = 0$.

$$\Rightarrow \bullet f'(x) = [(x+1)^{1/6}]' = \frac{1}{6}(x+1)^{-5/6} = \frac{1}{6\sqrt[6]{(x+1)^5}}, x > 0, f'(x_0) = f'(0) = \frac{1}{6}.$$

- Let $O(0)$ be such that $0,06 \in O(0)$.

$$\Rightarrow \bullet \sqrt[6]{x} = f(x) \approx f(0) + f'(0) \cdot x = 1 + \frac{x}{6} = \frac{x+6}{6}.$$

$$\Rightarrow \bullet \sqrt[6]{1,06} = f(0,06) \approx \frac{0,06+6}{6} = \frac{6,06}{6} = 1,01. \blacksquare$$

```

(%i8) c:0.06$ f(x):=(x+1)^(1/6)$
s:0$ f1(x):=diff(f(x),x,1)$
p(x):=f(s)+subst(s,x,f1(x))*(x-s)$ p(x);
h(c):=print("c=",c,"
c^(1/6)=",f(c),"=",float(f(c)),"approx",
subst(c,x,float(p(x))))$ h(c)$
(%O6) x
      6
+ 1
c = 0.06 (c + 1)^(1/6) = f(1.06) = 1.009758794179192 approx 1.01

```

A function $y = f(x)$, $x \in D(f)$ is differentiable, then (if they exist):

- $y = f'(x) = f^{(1)}(x)$, $x \in A_1 \subset D(f)$, $A_1 \neq \emptyset$.

First-order derivative (first derivative) of f on the set A_1 .

- $y = [f'(x)]' = f''(x) = f^{(2)}(x)$, $x \in A_2 \subset A_1$, $A_2 \neq \emptyset$.

The second-order derivative (second derivative) of f on the set A_2 .

- $y = [f''(x)]' = f'''(x) = f^{(3)}(x)$, $x \in A_3 \subset A_2$, $A_3 \neq \emptyset$.

Third-order derivative (third derivative) of f on the set A_3 .

...

- $y = [f^{(n-1)}(x)]' = f^{(n)}(x)$, $x \in A_n \subset A_{n-1}$, $A_n \neq \emptyset$.

The derivative of the n -th order (n -th derivative) of f on the set A_n .

Specially: • $y = f(x) = f^{(0)}(x)$, $x \in D(f) = A_0$.

Zero derivative (zero derivative) of f on the set A_0 .

$y = f(x)$, $x \in D(f)$ and $x_0 \in D(f)$, then the n -th derivative of f at the point x_0 (if it exists):

$$\bullet f^{(n)}(x_0) = \lim_{x \rightarrow x_0} \frac{f^{(n-1)}(x) - f^{(n-1)}(x_0)}{x - x_0} = \lim_{h \rightarrow 0} \frac{f^{(n-1)}(x_0+h) - f^{(n-1)}(x_0)}{h}, \quad x \in A_n, n \in \mathbb{N}.$$

The function $f^{(n-1)}$ must be defined in some neighborhood of $O(x_0)$.

Calculating $f^{(n)}$, $n \in \mathbb{N}$ can be very laborious in general.

Example.

$$y = x^k, \quad x \in \mathbb{R}, k \in \mathbb{N}.$$

$$[x^k]' = kx^{k-1}, \quad [x^k]'' = k(k-1)x^{k-2}, \quad [x^k]''' = k(k-1)(k-2)x^{k-3}, \dots,$$

$$[x^k]^{(k-1)} = k(k-1) \dots 2x, \quad [x^k]^{(k)} = k!, \quad [x^k]^{(k+1)} = 0, \dots$$

$$\bullet [x^k]^{(n)} = \begin{cases} k(k-1) \dots (k-n+1)x^{k-n}, & x \in \mathbb{R} \text{ for } n \in \mathbb{N}, n \leq k. \\ 0, & x \in \mathbb{R} \text{ for } n \in \mathbb{N}, n > k. \blacksquare \end{cases}$$

Example.

$$\bullet y = e^x, x \in \mathbb{R}. \implies \bullet [e^x]^{(n)} = e^x, x \in \mathbb{R} \text{ for } n \in \mathbb{N}. \blacksquare$$

Example.

$$\bullet y = \sin x, x \in \mathbb{R}, y = \cos x, x \in \mathbb{R}.$$

$$[\sin x]' = \cos x, \quad [\sin x]'' = [\cos x]' = -\sin x, \quad [\sin x]''' = [\cos x]'' = -\cos x,$$

$$[\sin x]^{(4)} = [\cos x]''' = \sin x, \quad [\sin x]^{(5)} = [\cos x]^{(4)} = \cos x, \dots$$

$$\bullet [\sin x]^{(n)} = [\sin x]^{(n+4)} = \begin{cases} (-1)^k \sin x, & x \in \mathbb{R} \text{ for } n = 2k, k \in \mathbb{N}. \\ (-1)^{k+1} \cos x, & x \in \mathbb{R} \text{ for } n = 2k - 1, k \in \mathbb{N}. \end{cases}$$

$$\bullet [\cos x]^{(n)} = [\cos x]^{(n+4)} = \begin{cases} (-1)^k \sin x, & x \in \mathbb{R} \text{ for } n = 2k, k \in \mathbb{N}. \\ (-1)^k \cos x, & x \in \mathbb{R} \text{ for } n = 2k - 1, k \in \mathbb{N}. \blacksquare \end{cases}$$

Leibniz formula.

Functions f, g have derivatives on the set A up to the order $n \in \mathbb{N}$ (including).

$$\implies \bullet [fg]^{(n)} = \sum_{i=0}^n \binom{n}{i} f^{(n-i)} g^{(i)} = \binom{n}{0} f^{(n)} g^{(0)} + \binom{n}{1} f^{(n-1)} g^{(1)} + \dots + \binom{n}{n} f^{(0)} g^{(n)}.$$

Applications of the derivative of a function

Theorems about the mean value of a function (Rolle's, Lagrange's) and l'Hospital's rule are among the most common applications of derivation in practice.

Rolle's mean value theorem.

A function f is continuous on $\langle a; b \rangle$.
 $f(a) = f(b)$.
 for all $x \in \langle a; b \rangle$: $f'(x) \in \mathbb{R}^*$.

$$\left. \begin{array}{l} \text{A function } f \text{ is continuous on } \langle a; b \rangle. \\ f(a) = f(b). \\ \text{for all } x \in \langle a; b \rangle: f'(x) \in \mathbb{R}^*. \end{array} \right\} \Rightarrow \bullet \text{ There exists } c \in \langle a; b \rangle: f'(c) = 0.$$

Lagrange's mean value theorem.

A function f is continuous on $\langle a; b \rangle$.
 for all $x \in \langle a; b \rangle$: $f'(x) \in \mathbb{R}^*$.

$$\left. \begin{array}{l} \text{A function } f \text{ is continuous on } \langle a; b \rangle. \\ \text{for all } x \in \langle a; b \rangle: f'(x) \in \mathbb{R}^*. \end{array} \right\} \Rightarrow \bullet \text{ There exists } c \in \langle a; b \rangle: f'(c) = \frac{f(b) - f(a)}{b - a}.$$

$$c \in \langle a; b \rangle. \Rightarrow \bullet c = a + \theta(b - a), \theta \in (0; 1).$$

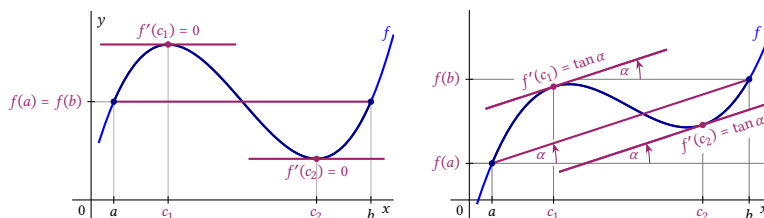
Let us denote $b = a + h$, $h \in \mathbb{R}$, for sufficiently small h we can assume $a + \theta h \approx a$.

$$\Rightarrow \bullet h = b - a, c = a + \theta(b - a) = a + \theta h, \theta \in (0; 1).$$

$$\bullet f(b) - f(a) = f(a + h) - f(a) = f'(a + \theta h) \cdot h, h \in \mathbb{R}, \theta \in (0; 1).$$

$$\bullet f(a + h) = f(a) + f'(a + \theta h) \cdot h \approx f(a) + f'(a)h = f(a) + df(a, h).$$

Rolle's and Lagrange's mean theorems guarantee the existence of $c \in \langle a; b \rangle$. However, we cannot find such points with them, nor can we determine their number.



Geometric meaning of Rolle's theorem (left) and Lagrange's theorem (right).

Indeterminate expressions of the type $\frac{0}{0}$, respectively $\frac{\infty}{\infty}$

are often they calculate using l'Hospital's rule.

L'Hospital's rule.

Functions f, g , point $a \in \mathbb{R}^*$.

- $\forall x \in O(a), x \neq a: f'(x) \in \mathbb{R}^*, g'(x) \in \mathbb{R}^*$.
 - There exists $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = b \in \mathbb{R}^*$.
 - $\lim_{x \rightarrow a} f(x) = \pm\infty, \lim_{x \rightarrow a} g(x) = \pm\infty$ ($L'H_{\infty}^{\infty}$).
 - resp. • $\lim_{x \rightarrow a} f(x) = 0, \lim_{x \rightarrow a} g(x) = 0$ ($L'H_{\frac{0}{0}}$).
- $$\left. \begin{array}{l} \bullet \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = b. \end{array} \right\} \Rightarrow$$

The existence of $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ does not imply the existence of $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$.

- It is very important to verify all the assumptions of l'Hospital's rule.
- Validity of the assumption $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = b \in \mathbb{R}^*$ is verified continuously during the limit calculation.

- We can also use L'Hospital's rule several times in a row:

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = \lim_{x \rightarrow a} \frac{f''(x)}{g''(x)} = \dots = \lim_{x \rightarrow a} \frac{f^{(k)}(x)}{g^{(k)}(x)}, k \in \mathbb{N}.$$

Example.

$$\lim_{x \rightarrow 2} \frac{x^3 - 8}{x - 2} = 12.$$

- $f(x) = x^3 - 8, x \in \mathbb{R}, g(x) = x - 2, x \in \mathbb{R}$.
- $O(2)$ can be chosen arbitrarily, e.g. $O(2) = \mathbb{R}$.

$$\left. \begin{array}{l} f'(x) = 3x^2, g'(x) = 1 \text{ for } x \in \mathbb{R} - \{2\}. \\ \lim_{x \rightarrow 2} \frac{f'(x)}{g'(x)} = \lim_{x \rightarrow 2} \frac{3x^2}{1} = 12. \\ \lim_{x \rightarrow 2} (x^3 - 8) = \lim_{x \rightarrow 2} (x - 2) = 0. \end{array} \right\} \Rightarrow \bullet \lim_{x \rightarrow 2} \frac{x^3 - 8}{x - 2} = \lim_{x \rightarrow 2} \frac{3x^2}{1} = 12. \blacksquare$$

```
(%i9) f(x) := (x^3 - 8) / (x - 2)
fc(x) := num(f(x)) $ fc(x);
fm(x) := denom(f(x)) $ fm(x);
'limit(f(x), x, 2); 'limit(diff(fc(x), x, 1) / diff(fm(x), x, 1), x, 2);
limit(f(x), x, 2); limit(diff(fc(x), x, 1) / diff(fm(x), x, 1), x, 2);
(%o4) x^3 - 8
(%o5) x - 2
(%o6) lim_{x -> 2} x^3 - 8 / x - 2
(%o7) 3 lim_{x -> 2} x^2
```

(%08) 12

(%09) 12

Example.

$$\bullet \lim_{x \rightarrow \infty} \frac{\ln x}{x} = [L'H_{\infty}^{\infty}] = \lim_{x \rightarrow \infty} \frac{[\ln x]'}{[x]'} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{1} = \lim_{x \rightarrow \infty} \frac{1}{x} = \frac{1}{\infty} = 0. \blacksquare$$

(%14) $f(x) := \log(x)/x$ $fc(x) := \text{num}(f(x))$ $fm(x) := \text{denom}(f(x))$

limit(diff(fc(x), x, 1)/diff(fm(x), x, 1), x, 2);

(%04) $\frac{1}{2}$ **Example.**

$$\bullet \lim_{x \rightarrow 0} \frac{x - \sin x}{x^3} = [L'H_0^0] = \lim_{x \rightarrow 0} \frac{1 - \cos x}{3x^2} = [L'H_0^0] = \lim_{x \rightarrow 0} \frac{0 - (-\sin x)}{6x} \\ = \lim_{x \rightarrow 0} \frac{\sin x}{6x} = [L'H_0^0] = \lim_{x \rightarrow 0} \frac{\cos x}{6} = \frac{1}{6}. \blacksquare$$

Example.

$$\bullet \lim_{x \rightarrow \infty} \frac{e^x + e^{-x}}{e^x - e^{-x}} = [L'H_{\infty}^{\infty}] = \lim_{x \rightarrow \infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = [L'H_{\infty}^{\infty}] = \lim_{x \rightarrow \infty} \frac{e^x + e^{-x}}{e^x - e^{-x}} = \dots$$

We cannot use L'Hospital's rule. \blacksquare

A function $y = f(x)$, $x \in D(f)$, neighborhood $O(x_0) \subset D(f)$, finite $f'(x_0)$, $f''(x_0)$, ..., $f^{(n)}(x_0) \in \mathbb{R}$.

Taylor polynomial of degree $n \in \mathbb{N}$ of the function f centered at the point $x_0 \in D(f)$ is defined as

$$T_n(x) = \sum_{k=0}^n \frac{f^{(k)}(x_0) \cdot (x-x_0)^k}{k!} \\ = f(x_0) + \frac{f'(x_0) \cdot (x-x_0)}{1!} + \frac{f''(x_0) \cdot (x-x_0)^2}{2!} + \dots + \frac{f^{(n)}(x_0) \cdot (x-x_0)^n}{n!}, \quad x \in O(x_0).$$

$T_n(x)$ is called **Maclaurin polynomial** for $x_0 = 0$:

$$T_n(x) = \sum_{k=0}^n \frac{f^{(k)}(0) \cdot x^k}{k!} = f(0) + \frac{f'(0) \cdot x}{1!} + \frac{f''(0) \cdot x^2}{2!} + \dots + \frac{f^{(n)}(0) \cdot x^n}{n!}, \quad x \in O(0).$$

Let us denote $h = x - x_0$, $x = x_0 + h$, $h \in O(0)$. \Rightarrow

$$T_n(x_0 + h) = \sum_{k=0}^n \frac{f^{(k)}(x_0) \cdot h^k}{k!} = f(x_0) + \frac{f'(x_0) \cdot h}{1!} + \frac{f''(x_0) \cdot h^2}{2!} + \dots + \frac{f^{(n)}(x_0) \cdot h^n}{n!}, \quad h \in O(0).$$

The remainder $R_n(x) = f(x) - T_n(x)$ expresses the approximation error of f using $T_n(x)$

$$R_n(x) = \frac{f^{(n+1)}(x_0 + \theta(x - x_0)) \cdot (x - x_0)^{n+1}}{(n+1)!}, \quad x \in O(x_0), \text{ where } \theta \in (0; 1). \quad (\text{Lagrangian form})$$

Approximation of f using $T_n(x)$ of degree $n \in \mathbb{N}$ in the center $x_0 \in D(f)$:

- It has a local character in the neighborhood of $O(x_0)$.
- It is the best of all approximations using degree n polynomials.

Example.

$$f(x) = \sqrt[3]{1+x} = (x+1)^{\frac{1}{3}}, \quad x \in \langle -1; \infty \rangle, \quad x_0 = 0. \quad \Rightarrow \quad f(0) = 1.$$

$$\bullet \quad f'(x) = \frac{1}{3}(x+1)^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{(x+1)^2}}, \quad x > -1. \quad \Rightarrow \quad f'(0) = \frac{1}{3}.$$

$$\bullet \quad f''(x) = -\frac{2}{3} \cdot \frac{1}{3}(x+1)^{-\frac{5}{3}} = \frac{-2}{9\sqrt[3]{(x+1)^5}}, \quad x > -1. \quad \Rightarrow \quad f''(0) = -\frac{2}{9}.$$

$$\bullet \quad f'''(x) = -\frac{5}{3} \cdot \left(-\frac{2}{9}\right) \cdot (x+1)^{-\frac{8}{3}} = \frac{10}{27\sqrt[3]{(x+1)^8}}, \quad x > -1. \quad \Rightarrow \quad f'''(0) = \frac{10}{27}.$$

$$\Rightarrow \bullet \quad T_3(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} = 1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81}, \quad x \in O(0).$$

$$\bullet \quad \sqrt[3]{1+x} \approx \begin{cases} 1 + \frac{x}{3}, & x \in O(0) \text{ with error } R_1(x). \\ 1 + \frac{x}{3} - \frac{x^2}{9}, & x \in O(0) \text{ with error } R_2(x). \\ 1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81}, & x \in O(0) \text{ with error } R_3(x). \blacksquare \end{cases}$$

We calculate the Taylor polynomial of the function $\sqrt{x^2+1}$. Manual derivation is quite laborious.

```
(%i1) f(x) := sqrt(x^2+1)$
```

```
(%i2) print("f(x)=", f(x), ", f'(x)=", diff(f(x), x), ",
```

```
      f''(x)=", ratsimp(diff(f(x), x, 2)), ",
```

```
      f'''(x)=", ratsimp(diff(f(x), x, 3)))$
```

$$f(x) = \sqrt{x^2+1}, \quad f'(x) = \frac{x}{\sqrt{x^2+1}}, \quad f''(x) = \frac{\sqrt{x^2+1}}{x^4+2x^2+1}, \quad f'''(x) = -\frac{3x\sqrt{x^2+1}}{x^6+3x^4+3x^2+1}$$

The polynomial `tp1` is of the ninth degree (practically eighth degree), therefore the output command `coeff(tp1, x, 10)` is the number 0. The polynomial `tp2` is of the tenth degree and the command output `coeff(tp2, x, 10)` is real coefficient $c_{10} = 7/256$.

```
(%i3) tp1:taylor(f(x),x,0,9);
(tp1) 1 +  $\frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{16} - \frac{5x^8}{128} + \dots$ 
(%i4) print("c_3=",coeff(tp1,x,3),"",c_4=",coeff(tp1,x,4),"",
           c_10=",coeff(tp1,x,10))$
c_3= 0, c_4=  $-\frac{1}{8}$ , c_10= 0
(%i5) tp2:taylor(f(x),x,0,10);
(tp2) 1 +  $\frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{16} - \frac{5x^8}{128} + \frac{7x^{10}}{256} + \dots$ 
(%i6) print("c_3=",coeff(tp2,x,3),"",c_4=",coeff(tp2,x,4),"",
           c_10=",coeff(tp2,x,10))$
c_3= 0, c_4=  $-\frac{1}{8}$ , c_10=  $\frac{7}{256}$ 
```

Example.

$f(x) = \ln x, x \in (0; \infty), x_0 = 1. \Rightarrow f(1) = 0.$

- $f'(x) = \frac{1}{x} = x^{-1}, x > 0. \Rightarrow f'(1) = 1 = 0!$
- $f''(x) = -\frac{1}{x^2} = -x^{-2}, x > 0. \Rightarrow f''(1) = -1 = -1!$
- $f'''(x) = 2\frac{1}{x^3} = 2x^{-3}, x > 0. \Rightarrow f'''(1) = 2 \cdot 1 = 2!$
- $f^{(4)}(x) = -3 \cdot 2\frac{1}{x^3} = -3 \cdot 2x^{-4}, x > 0. \Rightarrow f^{(4)}(1) = 3 \cdot 2 \cdot 1 = -3!$
- \dots
- $f^{(k)}(x) = (-1)^{k-1}(k-1)!\frac{1}{x^{k-1}}, x > 0, k \in \mathbb{N}. \Rightarrow f^{(k)}(1) = (-1)^{k-1}(k-1)!$

$$\Rightarrow \bullet T_n(x) = 0 + \sum_{k=1}^n \frac{f^{(k)}(1) \cdot (x-1)^k}{k!} = \sum_{k=1}^n \frac{(-1)^{k-1}(k-1)! \cdot (x-1)^k}{k!} = \sum_{k=1}^n \frac{(-1)^{k-1} \cdot (x-1)^k}{k}, x \in O(1). \blacksquare$$

```
(%i1) taylor(log(x),x,1,8);
(%o1) x - 1 -  $\frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} - \frac{(x-1)^6}{6} + \frac{(x-1)^7}{7} - \frac{(x-1)^8}{8} + \dots$ 
```

Sometimes it is more convenient to express $f(x) = \ln x$ in the form of a Maclaurin polynomial.

$$\bullet x = t + 1. \Rightarrow f(t) = \ln(t + 1), t \in (-1; \infty),$$

$$T_n(t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \frac{t^5}{5} + \dots + \frac{(-1)^{n-1}t^n}{n} = \sum_{k=1}^n \frac{(-1)^{k-1}t^k}{k}, t \in O(0).$$

```
(%i1) taylor(log(x+1), x, 0, 8);
```

```
(%O1) x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \frac{x^8}{8} + \dots
```

Example.

$f(x) = e^x, x \in \mathbb{R}. \Rightarrow$ The Maclaurin polynomial of degree $n \in \mathbb{N}$ has the form:

$$\bullet T_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^n}{n!} = \sum_{i=0}^n \frac{x^i}{i!}, x \in \mathbb{R}.$$

$f(x) = \sin x, x \in \mathbb{R}. \Rightarrow$ The Maclaurin polynomial of degree $n \in \mathbb{N}$ has the form:

$$\bullet T_{2k+1}(x) = 0 + \frac{x}{1!} + 0 + \frac{-x^3}{3!} + 0 + \frac{x^5}{5!} + 0 + \dots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} = \sum_{i=0}^k \frac{(-1)^i x^{2i+1}}{(2i+1)!}, x \in \mathbb{R}.$$

$f(x) = \cos x, x \in \mathbb{R}. \Rightarrow$ The Maclaurin polynomial of degree $n \in \mathbb{N}$ has the form:

$$\bullet T_{2k}(x) = 1 + 0 + \frac{-x^2}{2!} + 0 + \frac{x^4}{4!} + 0 + \dots + \frac{(-1)^k x^{2k}}{(2k)!} = \sum_{i=0}^k \frac{(-1)^i x^{2i}}{(2i)!}, x \in \mathbb{R}. \blacksquare$$

```
(%i1) taylor(exp(x), x, 0, 10);
```

```
(%O1) 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + \frac{x^8}{40320} + \frac{x^9}{362880} + \frac{x^{10}}{3628800} + \dots
```

```
(%i2) taylor(sin(x), x, 0, 10);
```

```
(%O2) x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{x^9}{362880} + \dots
```

```
(%i3) taylor(cos(x), x, 0, 10);
```

```
(%O3) 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320} - \frac{x^{10}}{3628800} + \dots
```

- We can approximate the functions $y = e^x, y = \sin x, y = \cos x$ for every $x \in \mathbb{R}$.
- We achieve the required accuracy by sufficiently increasing the degree of n .

Example.

$f(x) = e^{(x^2)}, x \in \mathbb{R}.$

Let us denote $g(t) = e^t, t \in \mathbb{R}, t = x^2. \Rightarrow f(x) = e^{(x^2)} = g(x^2) = g(t) = e^t.$

For the Maclaurin polynomial $P_n(t)$ of the function $g(t), t \geq 0$

and the Maclaurin polynomial $T_{2n}(x)$ of the function $f(x)$, $x \in \mathbb{R}$ holds:

$$\bullet P_n(t) = \sum_{i=0}^n \frac{t^i}{i!} = \sum_{i=0}^n \frac{(x^2)^j}{i!} = \sum_{i=0}^n \frac{x^{2i}}{i!} = 1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \dots + \frac{x^{2n}}{n!} = T_{2n}(x). \blacksquare$$

```
(%i1) taylor(exp(x^2), x, 0, 10);
(%o1) 1 + x^2 + x^4/2 + x^6/6 + x^8/24 + x^10/120 + ...
(%i3) subst(x^2, t, taylor(exp(t), t, 0, 5));
      subst(x^2, t, taylor(exp(t), t, 0, 10));
(%o2) x^10/120 + x^8/24 + x^6/6 + x^4/2 + x^2 + 1
(%o3) x^10/362880 + x^8/362880 + x^6/40320 + x^4/5040 + x^2/720 + x^10/120 + x^8/24 + x^6/6 + x^4/2 + x^2 + 1
```

At the end of this section, we find the Maclaurin polynomial of degree 10 of the function $f(x) = \ln \frac{x^2+1}{x+1}$.

```
(%i1) taylor(log((x^2+1)/(x+1)), x, 0, 10);
(%o1) -x + 3x^2/2 - x^3/3 - x^4/4 - x^5/5 + x^6/2 - x^7/7 - x^8/8 - x^9/9 + 3x^10/10 + ...
(%i3) tp1(x) := taylor(log(x^2+1), x, 0, 10) - taylor(log(x+1), x, 0, 10) $ tp1(x);
(%o3) -x + 3x^2/2 - x^3/3 - x^4/4 - x^5/5 + x^6/2 - x^7/7 - x^8/8 - x^9/9 + 3x^10/10 + ...
(%i6) tp2(x) := ratsimp(subst(x^2, t, taylor(log(t+1), t, 0, 5))
                        - taylor(log(x+1), x, 0, 10)) $
      tp2(x); tp1(x) - tp2(x);
(%o5) (756x^10 - 280x^9 - 315x^8 - 360x^7 + 1260x^6 - 504x^5 - 630x^4 - 840x^3 + 3780x^2 - 2520x) / 2520
(%o6) 0 + ...
```

Investigation of behaviour of functions

An important part of the investigation of the course of the function is the determination of the intervals, na for which this function is monotonic.

I is an interval, a function f is continuous on I , for all $x \in I$ there exists $f'(x) \in \mathbb{R}$ (finite).

The function f is on I :	• Increasing.	$\Leftrightarrow \forall x \in I$:	• $f'(x) > 0$.
	• Decreasing.	\Leftrightarrow	• $f'(x) < 0$.
	• Non-decreasing.	\Leftrightarrow	• $f'(x) \geq 0$.
	• Non-increasing.	\Leftrightarrow	• $f'(x) \leq 0$.
	• Constant.	\Leftrightarrow	• $f'(x) = 0$.

The points at which the continuous function f has local extrema, are closely related to the intervals on which this function is strictly monotonic.

A necessary condition for the existence of a local extremum.

$y = f(x)$, $x \in D(f)$, $x_0 \in D(f)$ is an interior point of $D(f)$, there exists $f'(x_0)$.

- The function f has a local extremum at the point x_0 . \implies • $f'(x_0) = 0$.

A local extremum of the function f can also be a point where the derivative does not exist.

$f'(x_0) = 0$ does not guarantee a local extremum of the function f at the point $x_0 \in D(f)$.
If $f'(x_0) = 0$ holds, then the point $x_0 \in D(f)$ is called **stationary**.

When searching for local extrema of the function f , we have to investigate:

- All points $x \in D(f)$ $\left\{ \begin{array}{l} \text{for which holds } f'(x) = 0. \\ \text{in which } f'(x) \text{ does not exist.} \end{array} \right.$

When searching for the global extrema of the function f , we must additionally investigate:

- All boundary points $x \in D(f)$.

A sufficient condition for the existence of a local extremum.

$y = f(x)$, $x \in D(f)$, $x_0 \in D(f)$, $f'(x_0) = 0$, for all $x \in O(x_0)$ there exists $f'(x)$.

- $f'(x) > 0$ for $x < x_0$ (increases on the left). $\left. \begin{array}{l} \\ \\ \end{array} \right\} \implies$ • $f(x_0)$ is a strict local maximum.
- $f'(x) < 0$ for $x > x_0$ (decreases on the right). $\left. \begin{array}{l} \\ \\ \end{array} \right\}$
- $f'(x) < 0$ for $x < x_0$ (decreases on the left). $\left. \begin{array}{l} \\ \\ \end{array} \right\} \implies$ • $f(x_0)$ is a strict local minimum.
- $f'(x) > 0$ for $x > x_0$ (increases on the right). $\left. \begin{array}{l} \\ \\ \end{array} \right\}$
- $f'(x) > 0$, respectively $f'(x) < 0$ for $x \neq x_0$. \implies • $f(x_0)$ is not extremum.

When investigating local extrema of a function, we can also use its second derivative.

$y = f(x)$, $x \in D(f)$, $x_0 \in D(f)$, $f'(x_0) = 0$, $f''(x_0) \in \mathbb{R}$, $f''(x_0) \neq 0$ (finite nonzero).

- $f''(x_0) < 0$. \implies • $f(x_0)$ is a strict local maximum.
- $f''(x_0) > 0$. \implies • $f(x_0)$ is a strict local minimum.

An important part of the investigation of the behaviour of the function is the determination of the intervals, na for which this function is convex or concave.

I is an interval, a function f is continuous on I , for all $x \in I$ there exists $f'(x) \in \mathbb{R}$ (finite).

f is on I :	• Convex.	\Leftrightarrow	f' is on I :	• Non-decreasing.
	• Concave.	\Leftrightarrow		• Non-increasing.
	• Strictly convex.	\Leftrightarrow		• Increasing.
	• Strictly concave.	\Leftrightarrow		• Decreasing.

I is an interval, f is continuous on I , for all $x \in I$ there exists $f''(x) \in \mathbb{R}$ (finite).

f is on I :	• Convex.	$\Leftrightarrow \forall x \in I$:	• $f''(x) > 0$.
	• Concave.	\Leftrightarrow	• $f''(x) < 0$.
	• Strictly convex.	\Leftrightarrow	• $f''(x) \geq 0$.
	• Strictly concave.	\Leftrightarrow	• $f''(x) \leq 0$.

When investigating the convexity and concavity of the function f , we must investigate:

- All points $x \in D(f)$ where the function f is continuous and
 - $\left\{ \begin{array}{l} \text{for which exists } f''(x) = 0. \\ \text{in which } f''(x) \text{ does not exist.} \end{array} \right.$

A necessary condition for the existence of an inflection point.

$y = f(x)$, $x \in D(f)$, $x_0 \in D(f)$ is an interior point of $D(f)$, there exists $f'(x_0)$.

- x_0 is the inflection point of the function f . \Rightarrow • $f''(x_0) = 0$.

The function f can also have an inflection at a point

where the second derivative does not exist.

$y = f(x)$, $x \in D(f)$, $x_0 \in D(f)$, $f'(x_0) \in \mathbb{R}$, for all $x \in O(x_0)$ there exists $f''(x)$.

- $f''(x) > 0$ for $x < x_0$ (left convex).
 $f''(x) < 0$ for $x > x_0$ (right concave). $\left. \vphantom{\begin{array}{l} f''(x) > 0 \\ f''(x) < 0 \end{array}} \right\} \Rightarrow$ • x_0 is the inflection point of f .
- $f''(x) < 0$ for $x < x_0$ (left concave).
 $f''(x) > 0$ for $x > x_0$ (right convex). $\left. \vphantom{\begin{array}{l} f''(x) < 0 \\ f''(x) > 0 \end{array}} \right\} \Rightarrow$ • x_0 is the inflection point of f .
- $f''(x) > 0$, resp. $f''(x) < 0$ for $x \neq x_0$. \Rightarrow • x_0 is not an inflection point of f .

When investigating the inflection of a function, we can also use its third derivative.

$y = f(x)$, $x \in D(f)$, $x_0 \in D(f)$, $f''(x_0) = 0$, $f'''(x_0) \in \mathbb{R}$

- $f'''(x_0) \neq 0$ (non-zero). \Rightarrow • x_0 is the inflection point of f .

We can generalize the previous results.

$$y = f(x), x \in D(f), \text{ point } x_0 \in D(f), n \in \mathbb{N}.$$

$$f'(x_0) = f''(x_0) = \dots = f^{(n-1)}(x_0) = 0, f^{(n)}(x_0) \neq 0.$$

- $n = 2k-1, k \in \mathbb{N}$ (odd). $\left\{ \begin{array}{l} \bullet f^{(n)}(x_0) > 0. \Rightarrow \bullet f \text{ is increasing at the point } x_0. \\ \bullet f^{(n)}(x_0) < 0. \Rightarrow \bullet f \text{ is decreasing at the point } x_0. \end{array} \right\} f(x_0) \text{ is not extremal.}$
- $n = 2k, k \in \mathbb{N}$ (even). $\left\{ \begin{array}{l} \bullet f^{(n)}(x_0) > 0. \Rightarrow \bullet f(x_0) \text{ is a strict local minimum.} \\ \bullet f^{(n)}(x_0) < 0. \Rightarrow \bullet f(x_0) \text{ is a strict local maximum.} \end{array} \right.$

$$y = f(x), x \in D(f), \text{ point } x_0 \in D(f), n \in \mathbb{N}.$$

$$f'(x_0) \in \mathbb{R}, f''(x_0) = \dots = f^{(n-1)}(x_0) = 0, f^{(n)}(x_0) \neq 0.$$

- $n = 2k+1, k \in \mathbb{N}$ (odd). $\bullet x_0$ is the inflection point of the function f .
- $n = 2k, k \in \mathbb{N}$ (even). $\left\{ \begin{array}{l} \bullet f^{(n)}(x_0) > 0. \Rightarrow \bullet f \text{ is strictly convex at the point } x_0. \\ \bullet f^{(n)}(x_0) < 0. \Rightarrow \bullet f \text{ is strictly concave at the point } x_0. \end{array} \right.$

Behaviour of functions

Investigating the course of the function f means determining:

- Domain $D(f)$, points and intervals of continuity and discontinuity.
- Evenness, oddness, periodicity, respectively other special properties.
- One-sided limits at discontinuity points, boundary points, and $\pm\infty$ points.
- Zero points; intervals on which f is positive and negative.
- f' , stationary points, local and global extrema; intervals on which f is increasing, decreasing and constant.
- f'' , inflection points; intervals on which f is convex and concave.
- Asymptotes without slope and asymptotes with slope.
- Codomain $H(f)$ and sketch the graph of the function.

The graph usually gives us the most vivid idea of the behaviour of the function. During its construction, we use all the data found. Of course, many times they are insufficient, so we have to supplement them with appropriately chosen functional values.

Example.

The behaviour of the function $f(x) = \frac{8(x-2)}{x^2} = \frac{8x-16}{x^2}$.

```
(%i1) f(x) := (8*x-16)/x^2;
```

```
(%o1) f(x) :=  $\frac{8x-16}{x^2}$ 
```

- $D(f) = R - \{0\} = (-\infty; 0) \cup (0; \infty)$.

Using the command `denom` (denominator) we find out when the denominator is zero.

```
(%i3) fm:denom(f(x));solve(fm=0,x);
```

```
(fm) x^2
```

```
(%o3) [x = 0]
```

- f is not periodic, f is not even, f is not odd.
- f is continuous on intervals $(-\infty; 0)$, $(0; \infty)$, is discontinuous at the point 0.
- $\lim_{x \rightarrow \pm\infty} f(x) = \lim_{x \rightarrow \pm\infty} \frac{8x-16}{x^2} = \lim_{x \rightarrow \pm\infty} \left(\frac{8}{x} - \frac{16}{x^2} \right) = \frac{8}{\pm\infty} - \frac{16}{\infty} = 0 - 0 = 0$.

```
(%i5) limit(f(x),x,minf);limit(f(x),x,inf);
```

```
(%o4) 0
```

```
(%o5) 0
```

- $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{8(x-2)}{x^2} = \frac{-16}{0^+} = -\infty$, $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{8(x-2)}{x^2} = \frac{-16}{0^+} = -\infty$.

```
(%i7) limit(f(x),x,0,minus);limit(f(x),x,0,plus);
```

```
(%o6) -∞
```

```
(%o7) -∞
```

- The point $x = 0$ is a non-removable discontinuity point of the II. type.
- $x = 0$ is an asymptote without a slope.
- $f(x) = \frac{8x-16}{x^2} = 0 \Leftrightarrow 8x - 16 = 0 \Leftrightarrow x = 2$.

Using the command `num` (numerator) we find out when the numerator is zero.

```
(%i9) fc:num(f(x));solve(fc=0,x);
```

```
(fc) 8x - 16
```

(%o9) $[x = 2]$

- $x = 2$ is the zero point of f . $\Rightarrow \begin{cases} f(x) < 0 \text{ for } x \in (-\infty; 0), \\ f(x) < 0 \text{ for } x \in (0; 2), \\ f(x) > 0 \text{ for } x \in (2; \infty). \end{cases}$

$f(2) = 0$, f is not defined at the point $x = 0$.

\Rightarrow The function f does not change the sign on intervals $(-\infty; 0)$, $(0; 2)$, $(2; \infty)$.

\Rightarrow Just select any point in the given intervals and verify its value.

(%i13) $f(2); f(-1); f(1); f(3);$

(%o10) 0

(%o11) -24

(%o12) -8

(%o13) $\frac{8}{9}$

- $f'(x) = \left[\frac{8x-16}{x^2} \right]' = \frac{8x^2 - (8x-16)2x}{x^4} = \frac{32x-8x^2}{x^4} = \frac{32-8x}{x^3}, x \in \mathbb{R}, x \neq 0.$

(%i15) $f1(x) := \text{diff}(f(x), x, 1) \text{ ratsimp}(f1(x));$

(%o15) $-\frac{8x-32}{x^3}$

- $f'(x) = \frac{32-8x}{x^3} = 0. \Leftrightarrow 32 - 8x = 0. \Leftrightarrow x = 4.$

(%i16) $\text{solve}(f1(x)=0, x);$

(%o16) $[x = 4]$

- f' is discontinuous at 0.

(%i18) $f1m:\text{denom}(\text{ratsimp}(f1(x))); \text{solve}(f1m=0, x);$

(f1m) x^3

(%o18) $[x = 0]$

- $x = 4$ is the zero point of f' . $\Rightarrow \begin{cases} f'(x) < 0, f \text{ is decreasing for } x \in (-\infty; 0), \\ f'(x) > 0, f \text{ is increasing for } x \in (0; 4), \\ f'(x) < 0, f \text{ is decreasing for } x \in (4; \infty). \end{cases}$

$f'(4) = 0$, f' is not defined at the point $x = 0$.

\Rightarrow The function f' does not change the sign on intervals $(-\infty; 0)$, $(0; 4)$, $(4; \infty)$.

\Rightarrow Just select any point in the given intervals and verify its value.

```
(%i22) subst(4,x,f1(x));subst(-1,x,f1(x));subst(1,x,f1(x));
      subst(5,x,f1(x));
```

```
(%o19) 0
```

```
(%o20) -40
```

```
(%o21) 24
```

```
(%o22) - $\frac{8}{125}$ 
```

- f has a local maximum at the point $x = 4$ and also a global maximum $f(4) = 1$.

```
(%i23) f(4);
```

```
(%o23) 1
```

- f has neither a local nor a global minimum.

- $f''(x) = \left[\frac{32-8x}{x^3} \right]' = \frac{-8x^3 - (32-8x)3x^2}{x^6} = \frac{16x^3 - 96x^2}{x^6} = \frac{16x-96}{x^4}, x \in \mathbb{R}, x \neq 0.$

```
(%i25) f2(x):=diff(f(x),x,2)$ ratsimp(f2(x));
```

```
(%o25)  $\frac{16x-96}{x^4}$ 
```

- $f''(x) = \frac{16x-96}{x^4} = 0. \Leftrightarrow 16x - 96 = 0. \Leftrightarrow x = 6.$

```
(%i26) solve(f2(x)=0,x);
```

```
(%o26) [x = 6]
```

- f'' is discontinuous at 0.

```
(%i28) f2m:denom(ratsimp(f2(x)));solve(f2m=0,x);
```

```
(f2m)  $x^4$ 
```

```
(%o28) [x = 0]
```

- $x = 6$ is the zero point of f'' . $\Rightarrow \begin{cases} f''(x) < 0, f \text{ is concave for } x \in (-\infty; 0), \\ f''(x) < 0, f \text{ is concave for } x \in (0; 6), \\ f''(x) > 0, f \text{ is convex for } x \in (6; \infty). \end{cases}$

$f'(6) = 0$, f'' is not defined at the point $x = 0$.

\Rightarrow The function f'' does not change the sign on intervals $(-\infty; 0)$, $(0; 6)$, $(6; \infty)$.

\Rightarrow Just select any point in the given intervals and verify its value.

```
(%i32) subst(6, x, f2(x)); subst(-1, x, f2(x)); subst(1, x, f2(x));
      subst(7, x, f2(x));
(%o29) 0
(%o30) -112
(%o31) -80
(%o32)  $\frac{16}{2401}$ 
```

- The point $x = 6$ is the inflection point of the function f .

```
(%i33) f(6);
(%o33)  $\frac{8}{9}$ 
```

- $k = \lim_{x \rightarrow \pm\infty} \frac{f(x)}{x} = \lim_{x \rightarrow \pm\infty} \frac{8x-16}{x^3} = \lim_{x \rightarrow \pm\infty} \left(\frac{8}{x^2} - \frac{16}{x^3} \right) = 0 - 0 = 0.$
 - $q = \lim_{x \rightarrow \pm\infty} [f(x) - kx] = \lim_{x \rightarrow \pm\infty} [f(x) - 0 \cdot x] = \lim_{x \rightarrow \pm\infty} f(x) = 0.$
- $\} \Rightarrow y = kx + q = 0.$

```
(%i35) km: limit(f(x)/x, x, minf); kp: limit(f(x)/x, x, inf);
(km) 0
(kp) 0
(%i37) qm: limit(f(x)-km*x, x, minf); qp: limit(f(x)-kp*x, x, inf);
(km) 0
(kp) 0
```

- $y = 0$ is an asymptote with a slope.
- $H(f) = (-\infty; 1)$.

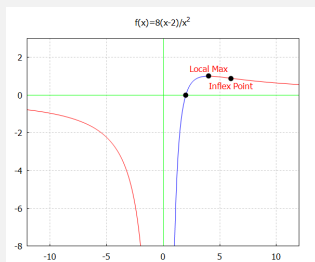
```
(%i38) draw2d(grid=true, xaxis=true, yaxis=true,
      xrange=[-12, 12], yrange=[-8, 3],
```



```

title="f(x)=8(x-2)/x^2",color=blue,explicit(f(x),x,0,4),
color=red,explicit(f(x),x,-12,0),explicit(f(x),x,4,12),
label(["Inflex Point",6,f(6)-.4],["Local Max",4,f(4)+.4],
color=green,parametric(0,t,t,-8,3),parametric(t,0,t,-12,12),
color=black,point_type=7,points([[4,f(4)], [6,f(6)], [2,f(2)]]))$

```



Chapter 4

The Indefinite Integral

Basic Terms

We motivated the introduction of the concept of derivative by the task of determining the instantaneous speed of a mass point, which moves in a straight line. We can reverse the problem and look for the trajectory of the mass point provided we know its instantaneous velocity at the given time.

$I \subset \mathbb{R}$ is an open interval, the function $F(x)$, $x \in I$ is called a **primitive function (antiderivative)** to the function $f(x)$, $x \in I$ on the interval I , if the derivative $F'(x)$ exists for all $x \in I$ and $F'(x) = f(x)$ holds for all $x \in I$.

$F(x)$ is primitive to $f(x)$ on the interval I , $c \in \mathbb{R}$ (constant).

$\Rightarrow G(x) = F(x) + c$ is primitive to the function $f(x)$ on the interval I .

It follows from the definition of a primitive function, that F is continuous on the interval I .

All primitive functions to a given function $f(x)$, $x \in I$ on the interval I differ from each other by a constant and form the set $\{F(x) + c, c \in \mathbb{R}\}$, (where F is any of the primitive functions). This set is called **indefinite integral of the function f on the interval I** and is denoted

$$\int f(x) dx = \{F(x) + c, x \in I, c \in \mathbb{R}\} = F(x) + c, x \in I, c \in \mathbb{R}.$$

$f(x)$, $x \in I$ is continuous on the interval I . \Rightarrow There exists $\int f(x) dx$.

The `integrate` command is used to integrate in the wxMaxima.

```
(%i1) 'integrate(1/(1+x^2),x)
```

```
(%o1)  $\int \frac{1}{x^2+1} dx$ 
```

```
(%i1) integrate(1/(1+x^2),x);
```

```
(%o1) atan x
```

```
(%i2) diff(%,x);
```

```
(%o2)  $\frac{1}{x^2+1}$ 
```

```
(%i1) f(x):=1/(1-x^2); integrate(f(x),x);
```

```
(%o1)  $\frac{1}{1-x^2}$ 
```

```
(%o2)  $\frac{\log(x+1)}{2} - \frac{\log(x-1)}{2}$ 
```

The table on the next page shows the basic formulas for integration. These formulas are closely related to formulas for derivatives of elementary functions and for practical needs **it is necessary to remember them.**

Methods of integration

Differentiation and integration are inverse operations on the interval I .

The function F is primitive to the function f on the interval I , $c \in R$, then for all $x \in I$:

$$\left[\int f(x) dx \right]' = [F(x) + c]' = f(x), \quad \int F'(x) dx = \int f(x) dx = F(x) + c.$$

Example.

- $\int \cot x dx = \int \frac{\cos x}{\sin x} dx = \int \frac{[\sin x]'}{\sin x} dx = \ln |\sin x| + c, x \in R - \{k\pi, k \in Z\}, c \in R.$

- $\int \tan x dx = \int \frac{\sin x}{\cos x} dx = - \int \frac{-\sin x}{\cos x} dx = - \int \frac{[\cos x]'}{\cos x} dx = - \ln |\cos x| + c,$

$$x \in R - \left\{ \frac{\pi}{2} + k\pi, k \in Z \right\}, c \in R.$$

Indefinite integrals of basic elementary functions

$\int dx = \int 1 dx = x + c, \quad x \in \mathbb{R}$	$\int x^a dx = \frac{x^{a+1}}{a+1} + c, \quad a \neq -1, x \neq 0$
$\int \frac{dx}{x} = \ln x + c, \quad x \neq 0$	$\int \frac{f'(x)}{f(x)} dx = \ln f(x) + c, \quad f(x) \neq 0, x \in D(f)$
$\int e^{ax} dx = \frac{e^{ax}}{a} + c, \quad a \neq 0, x \in \mathbb{R}$	$\int a^x dx = \frac{a^x}{\ln a} + c, \quad a > 0, a \neq 1, x \in \mathbb{R}$
$\int \sin ax dx = -\frac{\cos ax}{a} + c, \quad a \neq 0, x \in \mathbb{R}$	$\int \cos ax dx = \frac{\sin ax}{a} + c, \quad a \neq 0, x \in \mathbb{R}$
$\int \frac{dx}{\sin^2 ax} = -\frac{\cot ax}{a} + c, \quad a \neq 0, x \in \mathbb{R},$ $x \neq \frac{k\pi}{a}, k \in \mathbb{Z}$	$\int \frac{dx}{\cos^2 ax} = \frac{\tan ax}{a} + c, \quad a \neq 0, x \in \mathbb{R},$ $x \neq \frac{(2k+1)\pi}{2a}, k \in \mathbb{Z}$
$\int \sinh ax dx = \frac{\cosh ax}{a} + c, \quad a \neq 0, x \in \mathbb{R}$	$\int \cosh ax dx = \frac{\sinh ax}{a} + c, \quad a \neq 0, x \in \mathbb{R}$
$\int \frac{dx}{\sinh^2 ax} = -\frac{\coth ax}{a} + c, \quad a \neq 0, x \neq 0$	$\int \frac{dx}{\cosh^2 ax} = \frac{\tanh ax}{a} + c, \quad a \neq 0, x \in \mathbb{R}$
$\int \frac{dx}{x^2+a^2} = \frac{1}{a} \arctan \frac{x}{a} + c_1 = -\frac{1}{a} \operatorname{arccot} \frac{x}{a} + c_2, \quad a \neq 0, x \in \mathbb{R}$	
$\int \frac{dx}{x^2-a^2} = \int \frac{1}{2a} \left[\frac{1}{x-a} - \frac{1}{x+a} \right] dx = \frac{1}{2a} \ln \left \frac{x-a}{x+a} \right + c, \quad a \neq 0, x \in \mathbb{R} - \{a\}$	
$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin \frac{x}{ a } + c_1 = -\arccos \frac{x}{ a } + c_2, \quad a > 0, x \in (-a, a)$	
$\int \frac{dx}{\sqrt{x^2-a^2}} = \ln x + \sqrt{x^2-a^2} + c, \quad a > 0, x \in (-\infty, -a) \cup (a, \infty)$	
$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln(x + \sqrt{x^2+a^2}) + c, \quad a > 0, x \in \mathbb{R}$	

$$\bullet \int \sqrt[5]{x^3} dx = \int x^{\frac{3}{5}} dx = \frac{x^{\frac{3}{5}+1}}{\frac{3}{5}+1} + c = \frac{x^{\frac{8}{5}}}{\frac{8}{5}} + c = \frac{5}{8} x^{\frac{8}{5}} + c = \frac{5}{8} \sqrt[5]{x^8} + c, \quad x \geq 0, c \in \mathbb{R}.$$

$$\bullet \int |x| dx = \begin{cases} \int x dx = \frac{x^2}{2} + c = \frac{x \cdot x}{2} + c = \frac{x|x|}{2} + c & \text{for } x \geq 0, \\ \int (-x) dx = -\int x dx = -\frac{x^2}{2} + c = \frac{x \cdot (-x)}{2} + c = \frac{x|x|}{2} + c & \text{for } x < 0. \end{cases}$$

$$\Rightarrow \int |x| dx = \frac{x|x|}{2} + c, \quad x \in \mathbb{R}, c \in \mathbb{R}. \blacksquare$$

(%i1) integrate(cot(x), x);

(%o1) log(sin x)

(%i2) integrate(abs(x), x);

(%o2) $\frac{x|x|}{2}$

Decomposition method.

F, G are primitive functions to functions f, g on the interval $I, a, b \in \mathbb{R}, |a| + |b| > 0$.
 $\Rightarrow aF + bG$ is a primitive function to the function $af + bg$ on the interval I and holds

$$\int [af(x) + bg(x)] dx = a \int f(x) dx + b \int g(x) dx = aF(x) + bG(x) + c, \quad x \in I, c \in \mathbb{R}.$$

In the practical calculation, we do not write the intermediate expression

$$a \int f(x) dx + b \int g(x) dx$$

and we write it directly $\int [af(x) + bg(x)] dx = aF(x) + bG(x) + c$.

Example.

- $\int \frac{dx}{\sin^2 x \cos^2 x} = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx = \int \left[\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} \right] dx = \tan x - \cot x + c,$
 $x \in \mathbb{R}, x \neq \frac{k\pi}{2}, k \in \mathbb{Z}, c \in \mathbb{R}.$
- $\int \frac{(x-1)^2}{x} dx = \int \frac{x^2 - 2x + 1}{x} dx = \int \left[x - 2 + \frac{1}{x} \right] dx = \frac{x^2}{2} - 2x + \ln |x| + c,$
 $x \in \mathbb{R} - \{0\}, c \in \mathbb{R}.$
- $\int \tan^2 x dx = \int \frac{\sin^2 x}{\cos^2 x} dx = \int \frac{1 - \cos^2 x}{\cos^2 x} dx = \int \left[\frac{1}{\cos^2 x} - 1 \right] dx = \tan x - x + c,$
 $x \in \mathbb{R}, x \neq \frac{(2k+1)\pi}{2}, k \in \mathbb{Z}, c \in \mathbb{R}.$
- $\int \left[2 \cos x + x^3 + \frac{3}{x^2+1} \right] dx = 2 \sin x + \frac{x^4}{4} + 3 \arctan x + c, x \in \mathbb{R}, c \in \mathbb{R}. \blacksquare$

(%i1) integrate(1/(sin(x)^2*cos(x)^2), x);

(%o1) $\tan x - \frac{1}{\tan x}$

(%i2) integrate((x-1)^2/x, x);

(%o2) $\log x + \frac{x^2 - 4x}{2}$

Method per partes.

The functions u, v have continuous derivatives u', v' on the interval I .

$$\Rightarrow \int u(x) v'(x) dx = u(x) v(x) - \int u'(x) v(x) dx, \quad x \in I.$$

$$[uv]' = u'v + uv'. \Rightarrow uv = \int [uv]' = \int u'v + \int uv'. \Rightarrow \int uv' = uv - \int u'v.$$

Example.

- $\int x \cos x \, dx = \left[\begin{array}{l} u = x \\ v' = \cos x \end{array} \middle| \begin{array}{l} u' = 1 \\ v = \sin x \end{array} \right] = x \sin x - \int \sin x \, dx = x \sin x + \cos x + c, x \in R, c \in R.$
- $\int \ln x \, dx = \left[\begin{array}{l} u = \ln x \\ v' = 1 \end{array} \middle| \begin{array}{l} u' = \frac{1}{x} \\ v = x \end{array} \right] = x \ln x - \int dx = x \ln x - x + c, x \in (0; \infty), c \in R.$
- $\int \arctan x \, dx = \left[\begin{array}{l} u' = 1 \\ v = \arctan x \end{array} \middle| \begin{array}{l} u = x \\ v' = \frac{1}{1+x^2} \end{array} \right] = x \arctan x - \int \frac{x \, dx}{1+x^2} = x \arctan x - \frac{1}{2} \int \frac{0+2x}{1+x^2} \, dx$
 $= x \arctan x - \frac{1}{2} \ln |1+x^2| + c = x \arctan x - \ln \sqrt{1+x^2} + c, x \in R, c \in R. \blacksquare$

We can use the per partes method several times in a row, but we must be careful to stick to it they did not return to the original integral by reuse.

```
(%i1) u:x; v:integrate(sin(x),x);
(u) x
(v) -cos x
(%i3) u*v-integrate(v,x);
(%o3) sin x - x cos x
(%i4) integrate(x*sin(x),x);
(%o4) sin x - x cos x
```

Example.

Integral $I_n = \int x^n e^x \, dx, n = 0, 1, 2, 3, \dots$

- $I_n = \int x^n e^x \, dx = \left[\begin{array}{l} u = x^n \\ v' = e^x \end{array} \middle| \begin{array}{l} u' = nx^{n-1} \\ v = e^x \end{array} \right] = x^n e^x - n \int x^{n-1} e^x \, dx = x^n e^x - n I_{n-1}, n \in N.$

Specially for $n = 0, 1, 2, 3, x \in R, c \in R$ holds:

- $I_0 = \int x^0 e^x \, dx = \int e^x \, dx = e^x + c,$
- $I_1 = x e^x - I_0 = x e^x - e^x + c,$
- $I_2 = x^2 e^x - 2I_1 = x^2 e^x - 2[x e^x - e^x] + c = x^2 e^x - 2x e^x + 2 e^x + c,$
- $I_3 = x^3 e^x - 3I_2 = x^3 e^x - 3[x^2 e^x - 2x e^x + 2 e^x] + c = x^3 e^x - 3x^2 e^x + 6x e^x - 6 e^x + c. \blacksquare$

The per partes method is used quite often. It is suitable for integrating functions

$$P(x) e^{ax}, \quad P(x) \cos ax, \quad P(x) \sin ax, \quad P(x) \ln Q(x), \quad P(x) \arctan Q(x),$$

where $P(x), Q(x)$ are real polynomials, $a \in R, a \neq 0$.

Substitution method.

A function F is primitive to the function f on the interval I , $x = \varphi(t)$ has a derivative on the interval J , $\varphi(J) \subset I$.

$\Rightarrow F(\varphi(t))$ is primitive to the function $f(\varphi(t)) \cdot \varphi'(t)$ on J and holds

$$\int f(\varphi(t)) \cdot \varphi'(t) dt = \int f(x) dx = F(x) + c = F(\varphi(t)) + c, t \in J, c \in \mathbb{R}.$$

Sets I, J are intervals, $x = \varphi(t) : J \rightarrow I$ has a derivative $\varphi'(t) \neq 0$ on J , $F(t)$ is primitive to $f(\varphi(t)) \cdot \varphi'(t)$ on J .

$\Rightarrow F(\varphi^{-1}(x))$ is a primitive function $k f(x)$ on interval I and holds

$$\int f(x) dx = \int f(\varphi(t)) \cdot \varphi'(t) dt = F(t) + c = F(\varphi^{-1}(x)) + c, x \in I, c \in \mathbb{R}.$$

In the first case we do not have to use inverse substitution, but in the second case we have to use the inverse substitution $t = \varphi^{-1}(x)$.

Example.

$$\bullet \int \sin^3 t \cos t dt = \left[\begin{array}{l} \text{Subst. } x = \sin t \mid t \in \mathbb{R} \\ dx = \cos t dt \mid x \in (-1; 1) \end{array} \right] = \int x^3 dx = \frac{x^{3+1}}{3+1} + c = \frac{\sin^4 t}{4} + c, t \in \mathbb{R}, c \in \mathbb{R}.$$

$$\bullet \int \frac{f'(x)}{f(x)} dx = \left[\begin{array}{l} \text{Subst. } t = f(x) \\ dt = f'(x) dx \end{array} \right] = \int \frac{dt}{t} = \ln |t| + c = \ln |f(x)| + c, x \in D(f), c \in \mathbb{R}.$$

$$\bullet \int \frac{f'(t)}{f(t)} dt = \left[\begin{array}{l} \text{Subst. } x = f(t) \\ dx = f'(t) dt \end{array} \right] = \int \frac{dx}{x} = \ln |x| + c = \ln |f(t)| + c, t \in D(f), c \in \mathbb{R}.$$

$$\bullet \int \frac{x^2 dx}{x^3+1} = \left[\begin{array}{l} \text{Subst. } t = x^3 + 1 \mid x \in \mathbb{R} \\ dt = 3x^2 dx \mid t \in \mathbb{R} \end{array} \right] = \frac{1}{3} \int \frac{dt}{t} = \frac{1}{3} \ln |t| + c = \frac{1}{3} \ln |x^3 + 1| + c, x \in \mathbb{R}, c \in \mathbb{R}.$$

$$\bullet \int \frac{x^2 dx}{x^6+1} = \left[\begin{array}{l} \text{Subst. } t = x^3 \mid x \in \mathbb{R} \\ dt = 3x^2 dx \mid t \in \mathbb{R} \end{array} \right] = \frac{1}{3} \int \frac{dt}{t^2+1} = \frac{1}{3} \arctan t + c = \frac{1}{3} \arctan x^3 + c, x \in \mathbb{R}, c \in \mathbb{R}.$$

$$\bullet \int \frac{x^2 dx}{x^6-1} = \left[\begin{array}{l} \text{Subst. } t = x^3 \mid x \in \mathbb{R} - \{\pm 1\} \\ dt = 3x^2 dx \mid t \in \mathbb{R} - \{\pm 1\} \end{array} \right] = \frac{1}{3} \int \frac{dt}{t^2-1} = \frac{1}{3} \cdot \frac{1}{2} \ln \left| \frac{t-1}{t+1} \right| + c = \frac{1}{6} \ln \left| \frac{x^3-1}{x^3+1} \right|,$$

$x \in \mathbb{R} - \{\pm 1\}, c \in \mathbb{R}.$ ■

To perform a t -substitution, we:

- Decide on a substitution and use `diff` to produce the differential dt (called `de1(t)` in wx-Maxima), then express dx in terms of dt using `solve`.
- Extract the resulting equation using `%[1]` and replace `de1(x)` with its expression in terms of `de1(t)` in the integrand.
- Use `subst` to transform the entire integrand in terms of t , then perform the integral, remembering that `integrate` expects only the coefficient of `de1(t)`.
- Substitute the definition of t in terms of x into the resulting antiderivative.

(%i1)	INTEGRAND: (%e^(5*x))*diff(x);
(%o1)	e ^{5x} del(x)
(%i2)	solve(diff(t)=diff(5*x), del(x));
(%o2)	[del(x) = $\frac{\text{del}(t)}{5}$]
(%i3)	%[1];
(%o3)	del(x) = $\frac{\text{del}(t)}{5}$
(%i5)	subst(rhs(%), del(x), INTEGRAND)\$ subst(t, 5*x, %);
(%o5)	$\frac{e^t \text{del}(t)}{5}$
(%i6)	integrate(coeff(%), del(t), t);
(%o6)	$\frac{e^t}{5}$
(%i7)	subst(5*x, u, %);
(%o7)	$\frac{e^{5x}}{5}$
(%i8)	integrate(%e^(5*x), x);
(%o8)	$\frac{e^{5x}}{5}$

Example.

$f(x)$ has on the interval I a primitive function $F(x)$, $a, b \in R$, $a \neq 0$.

- $\int f(at + b) dt = \left[\text{Subst. } \begin{matrix} x = at + b \\ dx = a dt \end{matrix} \right] = \int \frac{f(x) dx}{a} = \frac{F(x)}{a} + c = \frac{F(at+b)}{a} + c.$
- $\int f(t + b) dt = \left[\text{Subst. } \begin{matrix} x = t + b \\ dx = dt \end{matrix} \right] = \int f(x) dx = F(x) + c = F(t + b) + c$ for $a = 1$.
- $\int f(-t) dt = \left[\text{Subst. } \begin{matrix} x = -t \\ dx = -dt \end{matrix} \right] = -\int f(x) dx = -F(x) + c = -F(-t) + c$ for $a = -1$. ■

Example.

- $\int \frac{dx}{\sqrt{1-x^2}} = \left[\text{Subst. } \begin{matrix} x = \sin t & | & x \in (-1; 1) \\ t = \arcsin x & | & t \in (-\frac{\pi}{2}; \frac{\pi}{2}) \end{matrix} \right] \begin{matrix} dx = \cos t dt, (\sin t)' = \cos t > 0 \text{ for } t \in (-\frac{\pi}{2}; \frac{\pi}{2}) \\ \sqrt{1-x^2} = \sqrt{1-\sin^2 t} = \sqrt{\cos^2 t} = |\cos t| = \cos t \end{matrix} \right] = \int \frac{\cos t dt}{\cos t} = \int dt = t + c = \arcsin x + c, x \in (-1; 1), c \in R.$
- $\int \frac{dx}{\sqrt{1-x^2}} = \left[\text{Subst. } \begin{matrix} x = \cos t & | & x \in (-1; 1) \\ t = \arccos x & | & t \in (0; \pi) \end{matrix} \right] \begin{matrix} dx = -\sin t dt, -(\cos t)' = \sin t > 0 \text{ for } t \in (0; \pi) \\ \sqrt{1-x^2} = \sqrt{1-\cos^2 t} = \sqrt{\sin^2 t} = |\sin t| = \sin t \end{matrix} \right] = \int \frac{-\sin t dt}{\sin t} = -\int dt = -t + c = -\arccos x + c, x \in (-1; 1), c \in R. \blacksquare$

Both solutions are correct because it holds for all $x \in \langle -1; 1 \rangle$ $\arcsin x + \arccos x = \frac{\pi}{2}$, i.e. $\arcsin x = -\arccos x + \frac{\pi}{2}$.

When integrating, different methods are often combined, and sometimes they have to be used several times in a row. If we use different integration procedures, we can arrive at other primitive functions.

Example.

$$\bullet \int \frac{\ln x}{x} dx = \left[\begin{array}{l} \text{Subst. } t = \ln x \mid x \in (0; \infty) \\ dt = \frac{dx}{x} \mid t \in R \end{array} \right] = \int t dt = \frac{t^2}{2} + c = \frac{\ln^2 x}{2} + c, x \in (0; \infty), c \in R.$$

$$\bullet \int \frac{\ln x}{x} dx = \left[\begin{array}{l} u = \ln x \mid u' = \frac{1}{x} \\ v' = \frac{1}{x} \mid v = \ln x \end{array} \right] = \ln^2 x - \int \frac{\ln x}{x} dx \\ \Rightarrow 2 \int \frac{\ln x}{x} dx = \ln^2 x + 2c \Rightarrow \int \frac{\ln x}{x} dx = \frac{\ln^2 x}{2} + c, x > 0, c \in R. \blacksquare$$

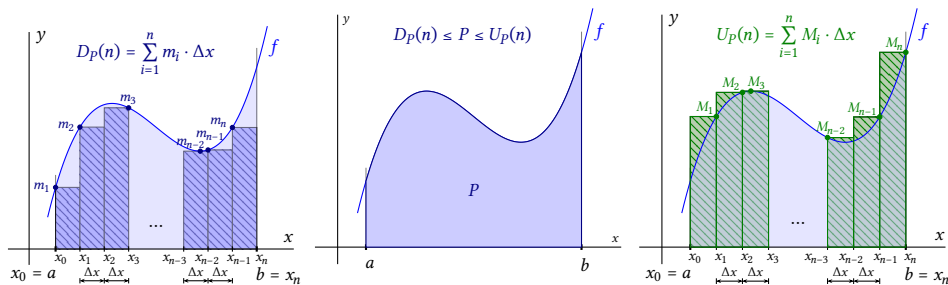
Chapter 5

The Definite Integral

Basic Terms

In this section, we will deal with the definite integral of the function, which, in contrast to the indefinite integral is not a function, but a specific value (a number or $\pm\infty$).

We can define a definite integral in several ways. We will define it using the so-called integral sums and call **Riemannian (definite) integral**.



The curvilinear trapezoid P determined by the non-negative function f on the interval $\langle a; b \rangle$ and its approximation using the sums D_p and U_p

Example.

$y = f(x)$, $x \in \langle a; b \rangle$ is a positive continuous function, $a, b \in R$, $a < b$.

Determine the area content of the set $P = \{[x; y] \in R^2, x \in \langle a; b \rangle, 0 \leq y \leq f(x)\}$.

- Let us divide $\langle a; b \rangle$ using points $a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$, $n \in N$ on n subintervals $\langle x_0; x_1 \rangle, \langle x_1; x_2 \rangle, \langle x_2; x_3 \rangle, \dots, \langle x_{n-2}; x_{n-1} \rangle, \langle x_{n-1}; x_n \rangle$ with the same length $\Delta x = x_1 - x_0 = x_2 - x_1 = \dots = x_n - x_{n-1} = \frac{b-a}{n}$.
- $m_i = \min \{f(x), x \in \langle x_{i-1}; x_i \rangle\}$, $M_i = \max \{f(x), x \in \langle x_{i-1}; x_i \rangle\}$ for $i = 1, 2, \dots, n$.

For the area P then:

$$\sum_{i=1}^n m_i \cdot \Delta x = L_p(n) \leq P \leq U_p(n) = \sum_{i=1}^n M_i \cdot \Delta x.$$

- If we decrease Δx (increase n), the estimates of L_P, U_P will improve (do not get worse).
- For $\Delta x = \frac{b-a}{n} \rightarrow 0$, i.e. $n \rightarrow \infty$ will hold $L_P(n) \rightarrow P, U_P(n) \rightarrow P$. ■

$\langle a; b \rangle$ is a non-degenerate interval (i.e., $a < b$). $y = f(x)$, $x \in \langle a; b \rangle$ is a bounded function, $D \in D_{\langle a; b \rangle}$, $D = \{x_i\}_{i=0}^n$, $n \in \mathbb{N}$.

- **Dividing the interval** $\langle a; b \rangle$ is called each finite set of points

$$D = \{x_0, x_1, x_2, \dots, x_n\} = \{x_i\}_{i=0}^n, \quad n \in \mathbb{N},$$

for which $a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$.

- x_0, x_1, \dots, x_n **dividing points** (they uniquely determine the dividing of D).
- Interval lengths $d_i = \langle x_{i-1}; x_i \rangle$, $i = 1, 2, \dots, n$ we denote $\Delta x_i = x_i - x_{i-1}$. We call the length of the longest of these intervals **dividing norm** D and denote $\mu(D)$, i.e. $\mu(D) = \max \{\Delta x_i, i = 1, 2, \dots, n\}$.
- For the sum of the lengths of the intervals d_1, d_2, \dots, d_n it holds

$$\Delta x_1 + \Delta x_2 + \dots + \Delta x_n = x_n - x_0 = b - a.$$

- The set of all dividings of the interval $\langle a; b \rangle$ we denote $D_{\langle a; b \rangle} = \{D, D \text{ is the division } \langle a; b \rangle\}$.
- $m_i = \inf \{f(x), x \in \langle x_{i-1}; x_i \rangle\}$, $M_i = \sup \{f(x), x \in \langle x_{i-1}; x_i \rangle\}$, $i = 1, 2, \dots, n$.
- **Lower** $S_L(f, D)$ and **upper Riemannian integral sum** $S_U(f, D)$ of function f under dividing D are called numbers

$$S_L(f, D) = \sum_{i=1}^n m_i \cdot \Delta x_i, \quad S_U(f, D) = \sum_{i=1}^n M_i \cdot \Delta x_i.$$

- Numbers

$$\underline{\int_a^b} f(x) dx = \sup \{S_L(f, D), D \in D_{\langle a; b \rangle}\}, \quad \overline{\int_a^b} f(x) dx = \inf \{S_U(f, D), D \in D_{\langle a; b \rangle}\}$$

we call the **lower and upper Riemannian (definite) integral of the function f on the interval $\langle a; b \rangle$ (from a to b)**. These numbers **always exist** and apply

$$m(b-a) \leq \underline{\int_a^b} f(x) dx \leq \overline{\int_a^b} f(x) dx \leq M(b-a),$$

while $m = \inf \{f(x), x \in \langle a; b \rangle\}$, $M = \sup \{f(x), x \in \langle a; b \rangle\}$.

- If the equality between the lower and upper Riemann integrals holds, then this common value

$$\underline{\int_a^b} f(x) dx = \overline{\int_a^b} f(x) dx = \int_a^b f(x) dx$$

we call the **Riemannian (definite) integral of the function f on the interval $\langle a; b \rangle$** . We call the function f **Riemannian integrable on the interval $\langle a; b \rangle$** and we denote $f \in R_{\langle a; b \rangle}$.

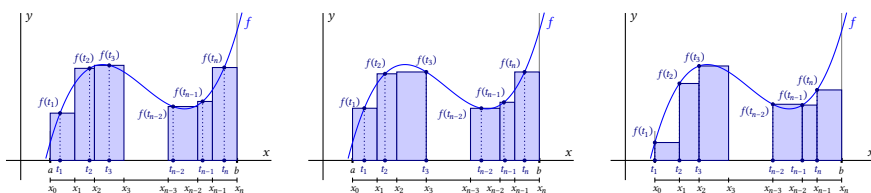
The integration variable has no effect on the value of the definite integral and instead of x we can write any symbol. This means that

$$\int_a^b f(x) dx = \int_a^b f(t) dt = \int_a^b f(u) du = \int_a^b f(z) dz = \int_a^b f(\varphi) d\varphi.$$

- **Riemannian (integral) sum** function f under dividing D and choice of points $T = \{t_1, t_2, \dots, t_n\} = \{t_i, t_i \in \langle x_{i-1}; x_i \rangle\}_{i=1}^n$ we call the number

$$S_T(f, D) = \sum_{i=1}^n f(t_i) \cdot \Delta x_i.$$

- The function f has infinitely many integral sums for a given dividing D . On the other hand, there are infinitely many dividing of the interval. If $y = f(x), x \in \langle a; b \rangle$ is continuous, then f takes its extrema on each interval $\langle x_{i-1}; x_i \rangle, i = 1, 2, \dots, n$ and $S_L(f, D)$ and $S_U(f, D)$ are also Riemann integral sums for some particular choices of points T .

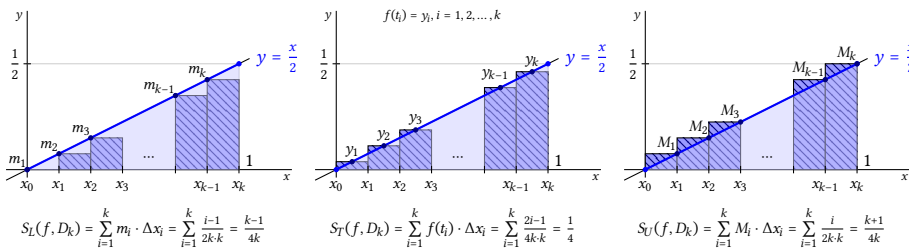


Integral sums of the function f under dividing D and different choices of points T

- In investigating Riemannian integrability function f on the interval $\langle a; b \rangle$ we do not need to deal with all by divisions $D \in D_{\langle a; b \rangle}$. It is sufficient to restrict ourselves to **normal sequences** of dividing $\{D_k\}_{k=1}^\infty \subset D_{\langle a; b \rangle}$, i.e. if $\lim_{k \rightarrow \infty} \mu(D_k) = 0$ holds.

Then it holds for every choice of points T

$$\lim_{k \rightarrow \infty} S_T(f, D_k) = \int_a^b f(x) dx.$$



$$S_L(f, D_k) = \sum_{i=1}^k m_i \cdot \Delta x_i = \sum_{i=1}^k \frac{i-1}{2k} \cdot \frac{1}{k} = \frac{k-1}{4k}$$

$$S_T(f, D_k) = \sum_{i=1}^k f(t_i) \cdot \Delta x_i = \sum_{i=1}^k \frac{2i-1}{4k} \cdot \frac{1}{k} = \frac{1}{4}$$

$$S_U(f, D_k) = \sum_{i=1}^k M_i \cdot \Delta x_i = \sum_{i=1}^k \frac{i}{2k} \cdot \frac{1}{k} = \frac{k+1}{4k}$$

Example.

$$\int_0^1 \frac{x}{2} dx = \frac{1}{4}.$$

- The function $f(x) = \frac{x}{2}, x \in \langle 0; 1 \rangle$ is increasing, continuous, $f \in R_{\langle 0; 1 \rangle}$.
- A normal sequence of dividings $\{D_k\}_{k=1}^\infty \subset D_{\langle 0; 1 \rangle}$, while $D_k = \left\{ \frac{i}{k} \right\}_{i=0}^k$ pre $k \in \mathbb{N}$.

- For $i = 1, 2, \dots, k$ holds $\Delta x_i = \frac{1}{k}$, $m_i = f(x_{i-1}) = \frac{i-1}{2k}$, $M_i = f(x_i) = \frac{i}{2k}$.
- $S_L(f, D_k) = \sum_{i=1}^k m_i \cdot \Delta x_i = \sum_{i=1}^k \frac{i-1}{2k} \cdot \frac{1}{k} = \frac{0+1+\dots+(k-1)}{2k^2} = \frac{\frac{(0+k-1)k}{2}}{2k^2} = \frac{k-1}{4k} = \frac{1}{4} - \frac{1}{4k}$.
- $S_U(f, D_k) = \sum_{i=1}^k M_i \cdot \Delta x_i = \sum_{i=1}^k \frac{i}{2k} \cdot \frac{1}{k} = \frac{1+2+\dots+k}{2k^2} = \frac{\frac{(1+k)k}{2}}{2k^2} = \frac{k+1}{4k} = \frac{1}{4} + \frac{1}{4k}$.

$$\Rightarrow \int_0^1 \frac{x}{2} dx = \lim_{k \rightarrow \infty} S_L(f, D_k) = \lim_{k \rightarrow \infty} S_U(f, D_k) = \lim_{k \rightarrow \infty} \left(\frac{1}{4} \pm \frac{1}{4k} \right) = \frac{1}{4}.$$

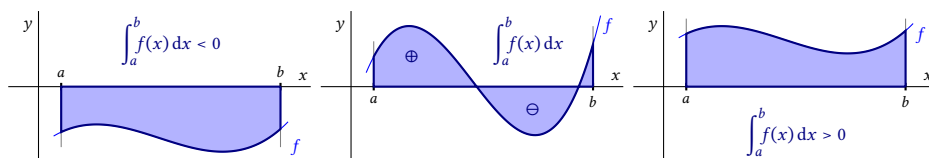
- Let's choose $T = \{t_i\}_{i=1}^k$ as points centers of intervals $\langle x_{i-1}; x_i \rangle$, $i = 1, 2, \dots, k$,
i.e. $t_i = \frac{1}{2} \left(\frac{i-1}{k} + \frac{i}{k} \right) = \frac{2i-1}{2k}$, then $f(t_i) = \frac{2i-1}{4k}$ and holds

$$S_T(f, D_k) = \sum_{i=1}^k f(t_i) \cdot \Delta x_i = \sum_{i=1}^k \frac{2i-1}{4k} \cdot \frac{1}{k} = \frac{1+3+\dots+(2k-1)}{4k^2} = \frac{\frac{(1+2k-1)k}{2}}{4k^2} = \frac{1}{4}.$$

$$\Rightarrow \int_0^1 \frac{x}{2} dx = \lim_{k \rightarrow \infty} S_T(f, D_k) = \lim_{k \rightarrow \infty} \frac{1}{4} = \frac{1}{4}. \blacksquare$$

Basic Properties

Geometrically, it represents the Riemannian definite integral on the interval $\langle a; b \rangle$ the area of the curvilinear trapezoid determined by the function f and the interval $\langle a; b \rangle$. Below the x axis (i.e., if f is negative), this area is negative.



Geometric representation of Riemann's definite integral

$f, g \in R_{\langle a; b \rangle}, c \in \mathbb{R} \Rightarrow cf \in R_{\langle a; b \rangle}, f + g \in R_{\langle a; b \rangle}$ and holds:

$$\int_a^b c f(x) dx = c \int_a^b f(x) dx, \quad \int_a^b [f(x) + g(x)] dx = \int_a^b f(x) dx + \int_a^b g(x) dx.$$

$f, g \in R_{\langle a; b \rangle} \Rightarrow |f|, f^2, fg \in R_{\langle a; b \rangle}.$

$f, g \in R_{\langle a; b \rangle}, \inf \{g(x), x \in \langle a; b \rangle\} > 0$, respectively $\sup \{g(x), x \in \langle a; b \rangle\} < 0$.

$$\Rightarrow \frac{1}{g}, \frac{f}{g} \in R_{\langle a; b \rangle}.$$

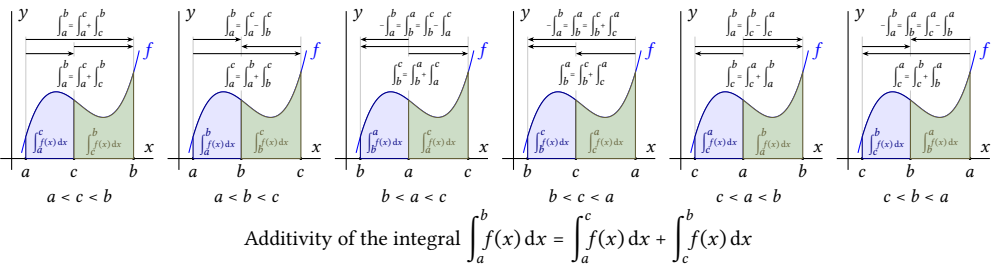
$$f \in R_{\langle a; b \rangle}, f(x) \geq 0 \text{ for all } x \in \langle a; b \rangle. \Rightarrow \int_a^b f(x) dx \geq 0.$$

$$f, g \in R_{\langle a; b \rangle}, g(x) \geq f(x) \text{ for all } x \in \langle a; b \rangle. \Rightarrow \int_a^b g(x) dx \geq \int_a^b f(x) dx.$$

Additivity of the integral.

A function $f \in R_I$, $I \subset \mathbb{R}$ is a bounded interval, points $a, b, c \in I$ are arbitrary.

$$\Rightarrow \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$



We can clearly illustrate the additivity of the Riemann integral on vectors. If we imagine the integrals $\int_a^b f(x) dx, \int_b^a f(x) dx = -\int_a^b f(x) dx$ as vectors $\vec{ab}, \vec{ba} = -\vec{ab}$ on the real axis, then e.g. for $a < b < c$ it holds

$$\int_a^b f(x) dx = \int_a^c f(x) + \int_c^b f(x) = \int_a^c f(x) - \int_b^c f(x),$$

respectively $\vec{ab} = \vec{ac} + \vec{cb} = \vec{ac} - \vec{bc}.$

Methods of integration

Calculation of the Riemann integral (Newton-Leibniz formula).

A function $f \in R_{\langle a; b \rangle}$, the function F is a primitive function to the function f on $\langle a; b \rangle$.

$$\Rightarrow \int_a^b f(x) dx = F(b) - F(a) = [F(x)]_a^b.$$

Example.

- $\int_{-1}^0 \frac{x}{2} dx = \left[\frac{x^2}{2 \cdot 2} \right]_{-1}^0 = \left[\frac{x^2}{4} \right]_{-1}^0 = \frac{0^2}{4} - \frac{(-1)^2}{4} = \frac{1}{4}.$
- $\int_{-1}^1 x^2 dx = \left[\frac{x^3}{3} \right]_{-1}^1 = \frac{1^2}{3} - \frac{(-1)^3}{3} = \frac{2}{3}.$
- $\int_0^1 \frac{dx}{\sqrt{x^2+1}} = \left[\ln \left(x + \sqrt{x^2+1} \right) \right]_0^1 = \ln(1 + \sqrt{2}) - \ln 1 = \ln(1 + \sqrt{2}).$
- $\int_{-1}^1 \frac{dx}{x} = \left[\ln|x| \right]_{-1}^1 = \ln 1 - \ln 1 = 0 \quad (?!?). \blacksquare$

```
(%i1) integrate(f(x), x, -1, 1);
```

```
(%o1)  $\int_{-1}^1 f(x) dx$ 
```

```
(%i3) f(x):=x^2$ F:integrate(f(x), x);
```

```
(F)  $\frac{x^3}{3}$ 
```

```
(%i4) integrate(f(x), x, -1, 1);
```

```
(%o4)  $\frac{2}{3}$ 
```

```
(%i5) subst(1, x, F)-subst(-1, x, F);
```

```
(%o5)  $\frac{2}{3}$ 
```

```
(%i6) float(subst(1, x, F)-subst(-1, x, F));
```

```
(%o6) 0.6666666666666666
```

```
(%i7) float(integrate(f(x), x, -1, 1));
```

```
(%o7) 0.6666666666666666
```

```
(%i2) f(x):=-cos(x)*sin(x)$ F:integrate(f(x), x);
```

```
(F)  $-\frac{\cos x^2}{2}$ 
```

```
(%i3) integrate(f(x), x, 1, 2);
```

(%O3)	$\frac{\cos 1^2}{2} - \frac{\cos 2^2}{2}$
(%i4)	subst(2,x,F)-subst(1,x,F);
(%O4)	$\frac{\cos 1^2}{2} - \frac{\cos 2^2}{2}$
(%i5)	float(integrate(f(x),x,1,2));
(%O5)	0.05937419607911741
(%i6)	float(subst(2,x,F)-subst(1,x,F));
(%O6)	0.05937419607911741

In the last integral $\int_{-1}^1 \frac{dx}{x}$ the problem is that the function $\frac{1}{x}$ (it is not defined at point 0) and the function $\ln x$ is not bounded on $\langle -1; 1 \rangle$ and in this sense we cannot calculate the integral.

(%i2)	f(x):=1/x\$ F:integrate(f(x),x);
(F)	-log x
(%i3)	integrate(f(x),x,-1,1);
	Principal Value
(%O3)	0
(%i4)	subst(1,x,F)-subst(-1,x,F);
(%O4)	-log(-1)

Definite integrals are generally calculated using indefinite integrals. We can modify the per partes method and substitution methods and calculate the definite integral using them directly. After substitution, we do not need to return to the original variables.

Method per partes.

$$u, v \in R_{\langle a; b \rangle}, u', v' \in R_{\langle a; b \rangle} \Rightarrow \int_a^b u(x) v'(x) dx = \left[u(x) v(x) \right]_a^b - \int_a^b u'(x) v(x) dx.$$

Example.

$$\begin{aligned} \int_0^{2\pi} x^2 \sin x dx &= \left[\begin{array}{l} u = x^2 \\ v' = \sin x \end{array} \middle| \begin{array}{l} u' = 2x \\ v = -\cos x \end{array} \right] = \left[-x^2 \cos x \right]_0^{2\pi} + \int_0^{2\pi} 2x \cos x dx \\ &= \left[\begin{array}{l} u = 2x \\ v' = \cos x \end{array} \middle| \begin{array}{l} u' = 2 \\ v = \sin x \end{array} \right] = \left[-4\pi^2 \cdot 1 + 0^2 \cdot 1 \right] + \left[2x \sin x \right]_0^{2\pi} - \int_0^{2\pi} 2 \sin x dx \\ &= -4\pi^2 + \left[4\pi \cdot 0 - 2 \cdot 0 \cdot 0 \right] - \left[-2 \cos x \right]_0^{2\pi} = -4\pi^2 - \left[-2 \cdot 1 + 2 \cdot 1 \right] = -4\pi^2. \blacksquare \end{aligned}$$

Substitution method.

I is an interval with boundaries a, b , J is an interval with boundaries α, β ,

$y = f(x)$ is continuous on I ,

$x = \varphi(t) : J \rightarrow R$, φ' is continuous on J , $\varphi(J) \subset I$, $\varphi(\alpha) = a$, $\varphi(\beta) = b$.

$$\Rightarrow f(\varphi)\varphi' \in R_J \text{ and holds } \int_a^b f(x) dx = \int_\alpha^\beta f[\varphi(t)]\varphi'(t) dt.$$

We can use the method in both directions.

Example.

$$\begin{aligned} \bullet \int_{-1}^1 \sqrt{1-x^2} dx & \left[\begin{array}{l} \text{Subst. } x = \sin t \mid x \in \langle -1; 1 \rangle \mid 1 = \sin \frac{\pi}{2} \mid \sqrt{1-x^2} = \sqrt{1-\sin^2 t} = \sqrt{\cos^2 t} = |\cos t| = \cos t \\ dx = \cos t dt \mid t \in \langle -\frac{\pi}{2}; \frac{\pi}{2} \rangle \mid -1 = \sin(-\frac{\pi}{2}) \mid \cos t \geq 0 \text{ pre v\v{s}etky } t \in \langle -\frac{\pi}{2}; \frac{\pi}{2} \rangle \end{array} \right] \\ & = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos 2t}{2} dt = \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [1 + \cos 2t] dt = \frac{1}{2} \left[t + \frac{\sin 2t}{2} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\ & = \frac{1}{2} \left[\frac{\pi}{2} + \frac{\sin \pi}{2} - \left(-\frac{\pi}{2} + \frac{\sin(-\pi)}{2} \right) \right] = \frac{1}{2} \left[\frac{\pi}{2} + 0 - \left(-\frac{\pi}{2} + 0 \right) \right] = \frac{1}{2} \cdot \pi = \frac{\pi}{2}. \\ \bullet \int_{-1}^2 t \sin(t^2 + 1) dt & = \left[\begin{array}{l} \text{Subst. } x = t^2 + 1 \mid t \in \langle -1; 0 \rangle \mid x \in \langle 1; 2 \rangle \mid t = 2 \mapsto x = 5 \\ dx = 2t dt \mid t \in \langle 0; 2 \rangle \mid x \in \langle 1; 5 \rangle \mid t = -1 \mapsto x = 2 \end{array} \right] = \frac{1}{2} \int_2^5 \sin x dx \\ & = \frac{1}{2} \left[-\cos x \right]_2^5 = \frac{1}{2} \left[-\cos 5 + \cos 2 \right] = \frac{\cos 2 - \cos 5}{2}. \blacksquare \end{aligned}$$

Integrating even and odd functions

A function $f \in R_{\langle a; b \rangle}$ is even or odd, where $a < b$. $\Rightarrow f(-x) \in R_{\langle -b; -a \rangle}$ and holds

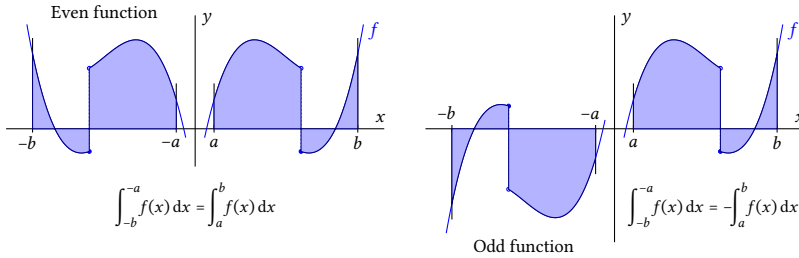
$$\int_a^b f(x) dx = \left[\begin{array}{l} \text{Subst. } t = -x \mid x = b \mapsto t = -b \\ dt = -dx \mid x = a \mapsto t = -a \end{array} \right] = - \int_{-a}^{-b} f(-t) dt = \int_{-b}^{-a} f(-t) dt = \int_{-b}^{-a} f(-x) dx.$$

$f \in R_{\langle a; b \rangle}$ is even, then

$$\int_a^b f(x) dx = \int_{-b}^{-a} f(-x) dx = \int_{-b}^{-a} f(x) dx.$$

$f \in R_{\langle a; b \rangle}$ is odd, then

$$\int_a^b f(x) dx = \int_{-b}^{-a} f(-x) dx = \int_{-b}^{-a} [-f(x)] dx = - \int_{-b}^{-a} f(x) dx.$$

**Example.**

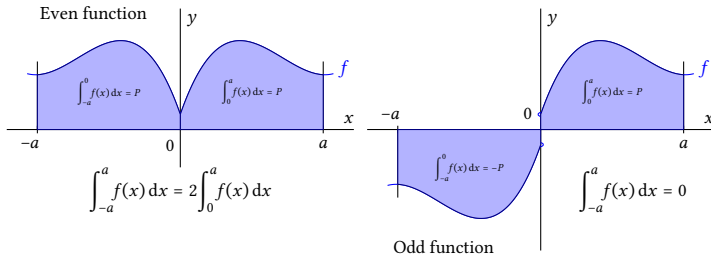
$a > 0$, $f \in R_{(-a;a)}$.

- f is even, then

$$\int_{-a}^a f(x) dx = \int_{-a}^0 f(x) dx + \int_0^a f(x) dx = \int_0^{-(a)} f(x) dx + \int_0^a f(x) dx = 2 \int_0^a f(x) dx.$$

- f is odd, then

$$\int_{-a}^a f(x) dx = \int_{-a}^0 f(x) dx + \int_0^a f(x) dx = - \int_0^{-(a)} f(x) dx + \int_0^a f(x) dx = 0. \blacksquare$$

**Example.**

- $\int_{-\pi}^{\pi} \sin |x| dx = \left[\begin{array}{l} \sin |x| \text{ is continuous} \\ \text{and even on } \langle -\pi; \pi \rangle \end{array} \right] = 2 \int_0^{\pi} \sin |x| dx = 2 \int_0^{\pi} \sin x dx = 2 \left[-\cos x \right]_0^{\pi} = -2 \cdot (-1) + 2 \cdot 1 = 4.$

- $\int_{-1}^1 \frac{x^3 \sqrt{x^2 + \sin^2 x}}{x^4 + 1} dx = \left[\begin{array}{l} \text{integrand is continuous} \\ \text{and odd function on } \langle -1; 1 \rangle \end{array} \right] = 0.$

- $\int_{-\pi}^{\pi} (x^4 - \sin^3 4x) dx = \int_{-\pi}^{\pi} x^4 dx - \int_{-\pi}^{\pi} \sin^3 4x dx = \left[\begin{array}{l} \text{function } x^4 \text{ is even on } \langle -\pi; \pi \rangle \\ \text{function } \sin^3 4x \text{ is odd on } \langle -\pi; \pi \rangle \end{array} \right] = 2 \int_0^{\pi} x^4 dx - 0 = 2 \left[\frac{x^5}{5} \right]_0^{\pi} = \frac{2\pi^5}{5}. \blacksquare$

A function $f \in R_{\langle a; b \rangle}$ is periodic with period $p > 0$,

$$f(x) = f(x + kp) \text{ for all } x \in \langle a; b \rangle, k \in \mathbb{Z}.$$

Substituting $x = \varphi(t) = t - kp$,

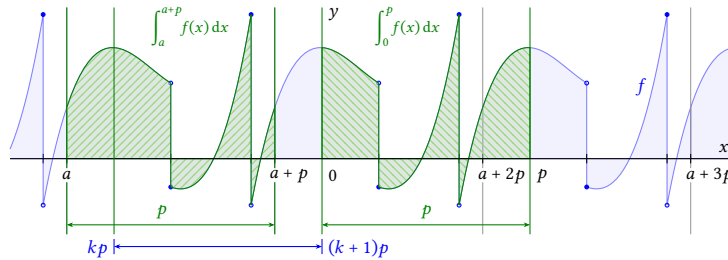
then $t = x + kp, t \in \langle a + kp; b + kp \rangle, dt = dx, f(x + kp) \in R_{\langle a+kp; b+kp \rangle}$ and holds

$$\int_a^b f(x) dx = \left[\begin{array}{l} x = b \mapsto t = b + kp \\ x = a \mapsto t = a + kp \end{array} \right] = \int_{a+kp}^{b+kp} f(t + kp) dt = \int_{a+kp}^{b+kp} f(t) dt = \int_{a+kp}^{b+kp} f(x) dx.$$

A function $y = f(x)$ is a periodic function with period $p > 0, a \in \mathbb{R}$. Then applies:

$$f \in R_{\langle 0; p \rangle} \Leftrightarrow f \in R_{\langle a; a+p \rangle} \text{ and (if they exist) } \int_0^p f(x) dx = \int_a^{a+p} f(x) dx.$$

The value of the Riemann integral of the periodic function f with period p is the same on each interval $\langle a; a + p \rangle, a \in \mathbb{R}$.



Example.

$a \in \mathbb{R}, m, n \in \mathbb{N}, m \neq n$.

$$\begin{aligned} \bullet \int_a^{a+2\pi} \sin^2(nx) dx &= \int_0^{2\pi} \sin^2(nx) dx = \int_0^{2\pi} \frac{1 - \cos(2nx)}{2} dx = \left[\frac{x}{2} - \frac{\sin(2nx)}{2 \cdot 2n} \right]_0^{2\pi} \\ &= \left[\frac{2\pi}{2} - \frac{\sin(2n \cdot 2\pi)}{4n} - \frac{0}{2} + \frac{\sin 0}{4n} \right] = [\pi - 0 - 0 + 0] = \pi. \end{aligned}$$

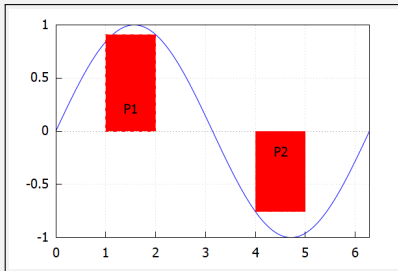
$$\begin{aligned} \bullet \int_a^{a+2\pi} \cos^2(nx) dx &= \int_0^{2\pi} \cos^2(nx) dx = \int_0^{2\pi} \frac{1 + \cos(2nx)}{2} dx = \left[\frac{x}{2} + \frac{\sin 2nx}{2 \cdot 2n} \right]_0^{2\pi} \\ &= \left[\frac{2\pi}{2} + \frac{\sin(2n \cdot 2\pi)}{4n} - \frac{0}{2} - \frac{\sin 0}{4n} \right] = [\pi + 0 - 0 - 0] = \pi. \end{aligned}$$

$$\begin{aligned} \bullet \int_a^{a+2\pi} \sin(mx) \sin(nx) dx &= \int_0^{2\pi} \sin(mx) \sin(nx) dx = \int_0^{2\pi} \frac{\cos(mx-nx) - \cos(mx+nx)}{2} dx \\ &= \left[\frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)} \right]_0^{2\pi} = \left[\frac{\sin(m-n)2\pi}{2(m-n)} - \frac{\sin(m+n)2\pi}{2(m+n)} - \frac{\sin 0}{2(m-n)} + \frac{\sin 0}{2(m+n)} \right] = 0. \end{aligned}$$

- $\int_a^{a+2\pi} \cos(mx) \cos(nx) dx = \int_0^{2\pi} \cos(mx) \cos(nx) dx = \int_0^{2\pi} \frac{\cos(mx-nx) + \cos(mx+nx)}{2} dx$
 $= \left[\frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)} \right]_0^{2\pi} = \left[\frac{\sin(m-n)2\pi}{2(m-n)} + \frac{\sin(m+n)2\pi}{2(m+n)} - \frac{\sin 0}{2(m-n)} - \frac{\sin 0}{2(m+n)} \right] = 0.$
- $\int_a^{a+2\pi} \sin(nx) \cos(nx) dx = \int_0^{2\pi} \sin(nx) \cos(nx) dx = \int_{-\pi}^{\pi} \sin(nx) \cos(nx) dx = 0.$
- $\int_a^{a+2\pi} \sin(mx) \cos(nx) dx = \int_0^{2\pi} \sin(mx) \cos(nx) dx = \int_{-\pi}^{\pi} \sin(mx) \cos(nx) dx = 0. \blacksquare$

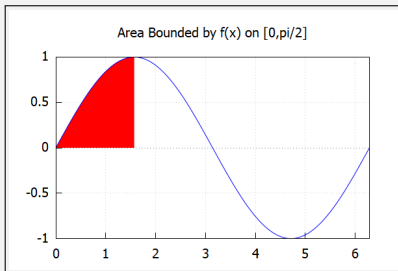
```
(%i1) f(x,n):=sin(n*x)^2;
(%o1) f(x,n) : sin(nx)^2
(%i2) integrate(f(x,n),x,0,2*%pi);
(%o2) -sin(4*%pi)-4*%pi
      4n
(%i3) integrate(f(x,n),x,a+0,a+2*%pi);
(%o3) sin(2an)-2an - sin((2a+4*%pi)n)+(-2a-4*%pi)n
      4n                4n
(%i4) ratsimp(%o3);
(%o4) -sin((2a+4*%pi)n)-sin(2an)-4*%pi
      4n
(%i5) integrate(f(x,4),x,0,2*%pi);
(%o5) pi
(%i6) integrate(f(x,4),x,a+0,a+2*%pi);
(%o6) sin(8a)-8a - sin(8a)-8a-16*%pi
      16                16
(%i7) ratsimp(%);
(%o8) pi
```

```
(%i1) f(x):=sin(x)$
wxdraw2d(grid=true,xaxis=true,yaxis=true,
color=blue, explicit(f(x),x,0,2*%pi),border=false,
rectangle([1,0],[2,f(2)]),color=black,label(["P1",1.5,0.2]),
rectangle([4,f(4)],[5,0]),color=black,label(["P2",4.5,-0.2]));
```

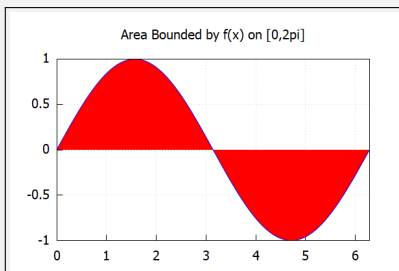


Recall that $\int_a^b f(x) dx$ gives us the area bounded between $f(x)$ and the x -axis.

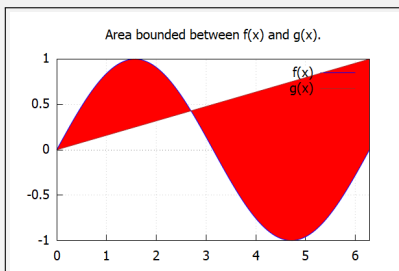
```
(%i1) f(x):=sin(x)$
wxdraw2d(grid=true,xrange=[0,2*%pi],yrange=[-1,1],
xaxis=true,yaxis=true,title="Area Bounded by f(x) on [0,pi/2]",
fill_color=red,filled_func=true,filled_func=f(x),
explicit(0,x,0,%pi/2),filled_func=false,
color=blue,explicit(f(x),x,0,2*%pi));
```



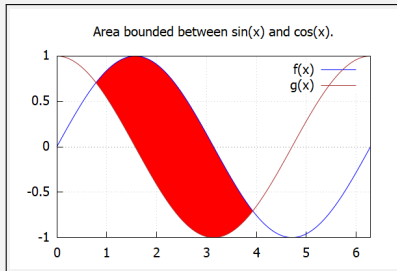
```
(%i1) f(x):=sin(x)$
wxdraw2d(grid=true,xrange=[0,2*%pi],yrange=[-1,1],
xaxis=true,yaxis=true,title="Area Bounded by f(x) on [0,2pi]",
fill_color=red,filled_func=true,filled_func=f(x),
explicit(0,x,0,2*%pi),filled_func=false,
color=blue,explicit(f(x),x,0,2*%pi));
```



```
(%i1) f(x):=sin(x)$ g(x):=x/(2*%pi)$
wxdraw2d(grid=true,xaxis=true,yaxis=true,
xrange=[0,2*%pi],yrange=[-1,1],
title="Area bounded between f(x) and g(x).",fill_color=red,
filled_func=true,filled_func=f(x),explicit(g(x),x,0,2*%pi),
filled_func=false,color=blue,key="f(x)",
explicit(f(x),x,0,2*%pi),color=brown,key="g(x)",
explicit(g(x),x,0,2*%pi));
```



```
(%i1) f(x):=sin(x)$ g(x):=cos(x)$
wxdraw2d(grid=true,xaxis=true,yaxis=true,
xrange=[0,2*%pi],yrange=[-1,1],
title="Area bounded between sin(x) and cos(x).",fill_color=red,
filled_func=true,filled_func=f(x),explicit(g(x),x,%pi/4,5*%pi/4),
filled_func=false,color=blue,key="f(x)",
explicit(f(x),x,0,2*%pi),color=brown,key="g(x)",
explicit(g(x),x,0,2*%pi));
```



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Žilinská univerzita v Žiline

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